5 Vectors

Let's Study

1 Vectors and their types.
2. Section formula.
3. Dot Product of Vectors.
4. Cross Product of Vectors.
5. Triple Product of Vectors.

Let's recall.

Scalar quantity : A quantity which can be completely described by magnitude only is called a scalar quantity. e.g. mass, length, temperature, area, volume, time distance, speed, work, money, voltage, density, resistance etc. In this book, scalars are given by real numbers.

Vector quantity : A quantity which needs to be described using both magnitude and direction is called a vector quantity. e.g. displacement, velocity, force, electric field, acceleration, momentum etc.

Let's learn.

5.1 Representation of Vector :

Vector is represented by a directed line segment.

If AB is a segment and its direction is shown with an arrowhead as in figure, then the directed segment AB has magnitude as well as direction. This is an example of vector.

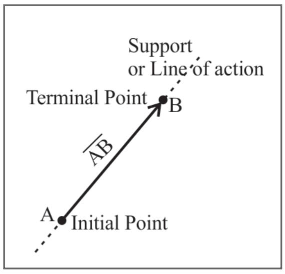


Fig. 5.1

The segment AB with direction from A to B denotes the vector $\overline{AB}$ read as $\overline{AB}$ read as 'AB bar' while direction from B to A denotes the vector $\overline{BA}$.

In vector $\overline{AB}$, the point A is called the initial point and the point B is called the terminal point

The directed line segment is a part of a line of unlimited length which is called the line of support or the line of action of the given vector.

If the initial and terminal point are not specified then the vectors are denoted by $\bar{a}$, $\bar{b}$, $\bar{c}$ or **a, b, c** (bold face) etc.

5.1.1 Magnitude of a Vector :

The magnitude (or size or length) of $\overline{AB}$ is denoted by $|\overline{AB}|$ and is defined as the length of segment AB. *i.e.* $|\overline{AB}| = l(AB)$

Magnitudes of vectors $\bar{a}, \bar{b}, \bar{c}$ are $|\bar{a}|, |\bar{b}|, |\bar{c}|$ respectively.

The magnitude of a vector does not depend on its direction. Since the length is never negative, $|\bar{a}| \geq 0$

5.1.2 Types of Vectors :

i) **Zero Vector :** A vector whose initial and terminal points coincide, is called a zero vector (or null vector) and denoted as $\bar{0}$. Zero vector cannot be assigned a definite direction and it has zero magnitude or it may be regarded as having any suitable direction. The vectors $\overline{AA}$, $\overline{BB}$ represent the zero vector and $|\overline{AA}| = 0$.

ii) **Unit Vector :** A vector whose magnitude is unity (*i.e.* 1 unity) is called a unit vector. The unit vector in the direction of a given vector $\bar{a}$ is denoted by $\hat{a}$, read as 'a-cap' or 'a-hat'.

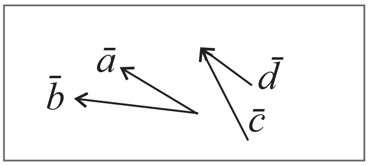


Fig. 5.2

iii) **Co-initial and Co-terminus Vectors :** Vectors having same initial point are called co-initial vectors, whereas vectors having same terminal point are called co-terminus vectors. Here $\bar{a}$ and are $\bar{b}$ co-initial vectors. $\bar{c}$ and $\bar{d}$ are co-terminus vectors.

iv) **Equal Vectors :** Two or more vectors are said to be equal vectors if they have same magnitude and direction.

- As $|\bar{a}| = |\bar{b}|$, and their directions are same regardless of initial point, we write as $\bar{a} = \bar{b}$.
- Here $|\bar{a}| = |\bar{c}|$, but directions are not same, so $\bar{a} \neq \bar{c}$.
- Here directions of $\bar{a}$ and $\bar{d}$ same but $|\bar{a}| \neq |\bar{d}|$, so $\bar{a} \neq \bar{d}$.

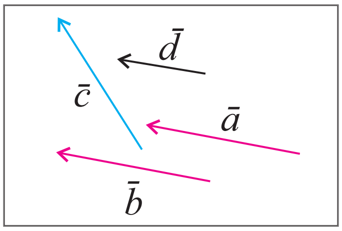


Fig 5.3

v) **Negative of a Vector :** If $\bar{a}$ is a given vector, then the negative of $\bar{a}$ is vector whose magnitude is same as that of $\bar{a}$ but whose direction is opposite to that of $\bar{a}$. It is denoted by $-\bar{a}$.

Thus, if $\overline{PQ} = \bar{a}$, then $\overline{QP} = -\bar{a} = -\overline{PQ}$.

Here $|\overline{PQ}| = |-\overline{QP}|$.

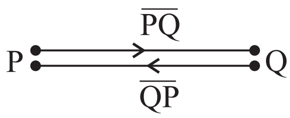


Fig. 5.4

vi) **Collinear Vectors :** Vectors are said to be collinear vectors if they are parallel to same line or they are along the same line.

vii) **Free Vectors :** If a vector can be translated anywhere in the space without changing its magnitude and direction then such a vector is called free vector. In other words, the initial point of free vector can be taken anywhere in the space keeping magnitude and direction same.

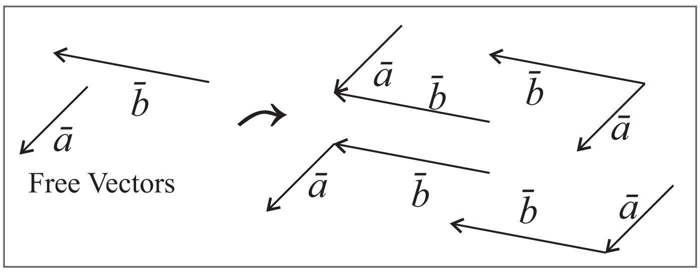


Fig. 5.5

viii) **Localised Vectors :** For a vector of given magnitude and direction, if it's initial point is fixed in space, then such a vector is called localised vector.

For example, if there are two stationary cars A and B on the road and a force is applied to car A, it is a localised vector and only car A moves, while car B is not affected.

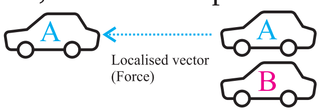


Fig 5.6

Note that in this chapter vectors are treated as free vectors unless otherwise stated.

Activity 1 :

Write the following vectors in terms of vectors $\bar{p}$, $\bar{q}$ and $\bar{r}$.

i) $\overrightarrow{AB} = \square$ ii) $\overrightarrow{BA} = \square$ iii) $\overrightarrow{BC} = \square$

iv) $\overrightarrow{CB} = \square$ v) $\overrightarrow{CA} = \square$ vi) $\overrightarrow{AC} = \square$

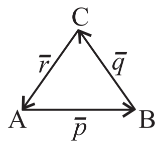


Fig 5.7

Algebra of Vectors :

5.1.3 Scalar Multiplication :

$2\bar{a}$ has the same direction as $\bar{a}$ but is twice as long as $\bar{a}$.

Let $\bar{a}$ be any vector and k be a scalar, then vector $k\,\bar{a}$, the scalar multiple vector of $\bar{a}$ is defined a vector whose magnitude is $|k\bar{a}| = |k|\,|\bar{a}|$ and vectors $\bar{a}$ and $k\,\bar{a}$ have the same direction if $k > 0$ and opposite direction if k < 0.

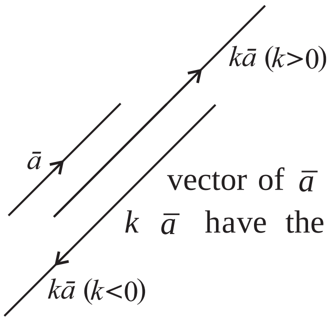


Fig 5.8

Note :

i) If $k = 0$, then $k\bar{a} = \bar{0}$.

ii) $\bar{a}$ and $k\bar{a}$ are collinear or parallel vectors.

iii) Two non zero vectors $\bar{a}$ and $\bar{b}$ are collinear or parallel if $\bar{a} = m\bar{b}$, where $m \neq 0$.

iv) Let $\hat{a}$ be the unit vector along non-zero vector $\bar{a}$ then $\bar{a} = |\bar{a}|\,\hat{a}$ or $\dfrac{\bar{a}}{|\bar{a}|} = \hat{a}$.

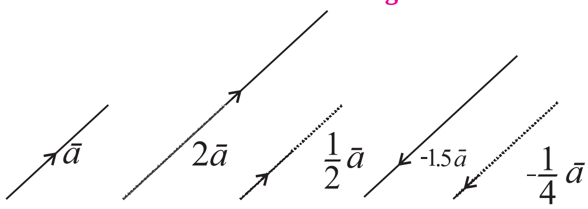


Fig 5.9

v) A vector of length k in the same direction as $\bar{a}$ is $k\hat{a} = k\left(\dfrac{\bar{a}}{|\bar{a}|}\right)$.

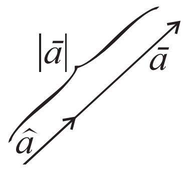


Fig 5.10

Now, consider a boat in a river going from one bank of the river to the other in a direction perpendicular to the flow of the river. Then, it is acted upon by two velocity vectors, one is the velocity imparted to the boat by its engine and other one is the velocity of the flow of river water. Under the simultaneous influence of these two velocities, the boat starts travelling with a different velocity. To have a precise velocity (i.e. resultant velocity) of the boat we use the law of addition of vectors.

5.1.4 Addition of Two Vectors : If $\bar{a}$ and $\bar{b}$ are any two vectors then their addition (or resultant) is denoted by $\bar{a} + \bar{b}$.

There are two laws of addition of two vectors.

Parallelogram Law : Let $\bar{a}$ and $\bar{b}$ be two vectors. Consider $\overline{AB}$ and $\overline{AD}$ along two adjacent sides of a parallelogram, such that $\overline{AB} = \bar{a}$ and $\overline{AD} = \bar{b}$ then $\bar{a} + \bar{b}$ lies along the diagonal of a parallelogram with $\bar{a}$ and $\bar{b}$ as sides.

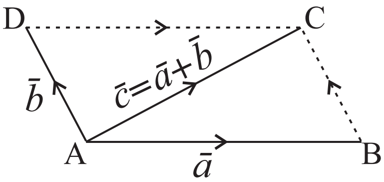


Fig 5.11

$\overline{AC} = \overline{AB} + \overline{AD}$ i.e. $\bar{c} = \bar{a} + \bar{b}$.

Triangle Law of addition of two vectors : Let $\bar{a}$, $\bar{b}$ be any two vectors then consider triangle ABC as shown in figure such that $\overline{AB} = \bar{a}$ and $\overline{BC} = \bar{b}$ then $\bar{a} + \bar{b}$ is given by vector $\overline{AC}$ along the third side of triangle ABC.

Thus, $\bar{a} + \bar{b} = \overline{AB} + \overline{BC} = \overline{AC}$.

This is known as the triangle law of addition of two vectors $\bar{a}$ and $\bar{b}$. The triangle law can also be applied to the ΔADC.

Here, $\overline{AD} = \overline{BC} = \bar{b}$, $\overline{DC} = \overline{AB} = \bar{a}$

Hence, $\overline{AD} + \overline{DC} = \bar{b} + \bar{a} = \overline{AC}$

Thus, $\overline{AC} = \bar{b} + \bar{a} = \bar{a} + \bar{b}$

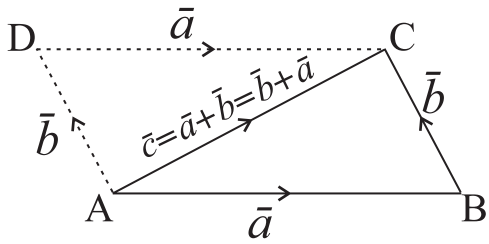


Fig 5.12

5.1.5 Subtraction of two vectors : If $\bar{a}$ and $\bar{b}$ are two vectors, then $\bar{a} - \bar{b} = \bar{a} + (-\bar{b})$, where $-\bar{b}$ is the negative vector of vector $\bar{b}$. Let $\overline{AB} = \bar{a}$, $\overline{BC} = \bar{b}$, now construct a vector $\overline{BD}$ such that its magnitude is same as the vector $\overline{BC}$, but the direction is opposite to that of it.

i.e. $\overline{BD} = -\overline{BC}$.

$\therefore \overline{BD} = -\bar{b}$.

Thus applying triangle law of addition.

We have

$\therefore \overline{AD} = \overline{AB} + \overline{BD}$

$= \overline{AB} - \overline{BC}$

$= \bar{a} - \bar{b}$

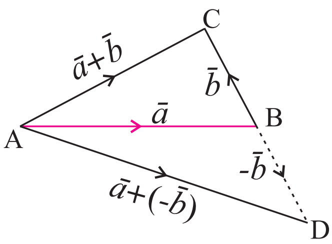


Fig 5.13

Note :

i) If (velocity) vectors $\bar{a}$ and $\bar{b}$ are acting simultaneously then we use parallelogram law of addition.

ii) If (velocity) vectors $\bar{a}$ and $\bar{b}$ are acting one after another then we use triangle law of addition.

iii) Adding vector to its opposite vector gives $\bar{0}$

As $\overline{PQ} + \overline{QP} = \overline{PP} = \bar{0}$ or

As $\overline{PQ} = -\overline{QP}$, then $\overline{PQ} + \overline{QP} = -\overline{QP} + \overline{QP} = \bar{0}$.

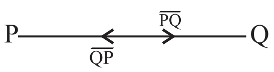


Fig 5.14

iv) In ΔABC, $\overline{AC} = -\overline{CA}$, so $\overline{AB} + \overline{BC} + \overline{CA} = \overline{AA} = \bar{0}$. This means that when the vectors along the sides of a triangle are in order, their resultant is zero as initial and terminal points become same.

v) The addition law of vectors can be extended to a polygon :

Let $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ be four vectors. Let $\overline{PQ} = \bar{a}$, $\overline{QR} = \bar{b}$, $\overline{RS} = \bar{c}$ and $\overline{ST} = \bar{d}$.

$\therefore\ \bar{a} + \bar{b} + \bar{c} + \bar{d}$

$= \overline{PQ} + \overline{QR} + \overline{RS} + \overline{ST}$

$= \left(\overline{PQ} + \overline{QR}\right) + \overline{RS} + \overline{ST}$

$= \left(\overline{PR} + \overline{RS}\right) + \overline{ST}$

$= \overline{PS} + \overline{ST}$

$= \overline{PT}$

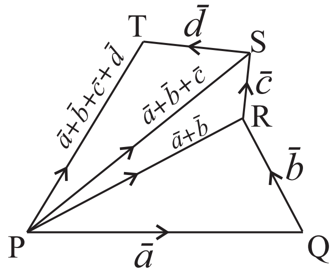


Fig 5.15

Thus, the vector $\overline{PT}$ represents sum of all vectors $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$.

This is also called as extended law of addition of vectors or polygonal law of addition of vectors.

vi) $\bar{a} + \bar{b} = \bar{b} + \bar{a}$ (commutative)

vii) $\left(\bar{a} + \bar{b}\right) + \bar{c} = \bar{a} + \left(\bar{b} + \bar{c}\right)$ (associative)

viii) $\bar{a} + \bar{0} = \bar{a}$ ($\bar{0}$ is additive identify)

ix) $\bar{a} + (-\bar{a}) = \bar{0}$ ($-\bar{a}$ is additive inverse)

x) If $\bar{a}$ and $\bar{b}$ are vectors and m and n are scalars, then

  i) $(m + n)\,\bar{a} = m\bar{a} + n\bar{a}$ (distributive)

  ii) $m(\bar{a} + \bar{b}) = m\bar{a} + m\bar{b}$ (distributive)

  iii) $m(n\bar{a}) = (mn)\bar{a} = n(m\bar{a})$

xi) $\left|\bar{a} + \bar{b}\right| \le \left|\bar{a}\right| + \left|\bar{b}\right|$, this is known as "Triangle Inequality".

This is obtained from the triangle law, as the length of any side of triangle is less than the sum of the other two sides. *i.e.* in triangle ABC, AC < AB + BC,

where, AC = $\left|\bar{a} + \bar{b}\right|$, $AB = \left|\bar{a}\right|$, $BC = \left|\bar{b}\right|$

xii) Any two vectors $\bar{a}$ and $\bar{b}$ determine a plane and vectors $\bar{a} + \bar{b}$ and $\bar{a} - \bar{b}$ lie in the same plane.

Activity 2 :

In quadrilateral PQRS, find a resultant vector

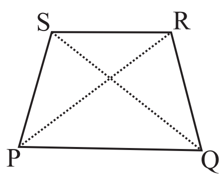


Fig 5.16

i) $\overline{QR} + \overline{RS} = \square$

ii) $\overline{PQ} + \overline{QR} = \square$

iii) $\overline{PS} + \overline{SR} + \overline{RQ} = \square$

iv) $\overline{PR} + \overline{RQ} + \overline{QS} = \square$

v) $\overline{QR} - \overline{SR} - \overline{PS} = \square$

vi) $\overline{QP} - \overline{RP} + \overline{RS} = \square$

Theorem 1:

Two non-zero vectors $\bar{a}$ and $\bar{b}$ are collinear if and only if there exist scalars m and n, at least one of them is non-zero such that $m\bar{a} + n\bar{b} = \bar{0}$.

Proof : Only If - part :

Suppose $\bar{a}$ and $\bar{b}$ are collinear.

$\therefore$ There exists a scalar $t \neq 0$ such that $\bar{a} = t\bar{b}$

$\therefore \bar{a} - t\bar{b} = \bar{0}$

i.e. $m\bar{a} + n\bar{b} = \bar{0}$, where $m = 1$ and $n = -t$.

If - part :

Conversely, suppose $m\bar{a} + n\bar{b} = \bar{0}$ and $m \neq 0$.

$\therefore m\bar{a} = -n\bar{b}$

$\therefore \bar{a} = \left(-\dfrac{n}{m}\right)\bar{b}$, where $t = \left(-\dfrac{n}{m}\right)$ is a scalar.

i.e. $\bar{a} = t\bar{b}$,

$\therefore \bar{a}$ is scalar multiple of $\bar{b}$.

$\therefore \bar{a}$ and $\bar{b}$ are collinear.

Corollary 1 : If two vectors $\bar{a}$ and $\bar{b}$ are not collinear and $m\bar{a} + n\bar{b} = \bar{0}$, then $m = 0$, $n = 0$. (This can be proved by contradiction assuming $m \neq 0$ or $n \neq 0$).

Corollary 2 : If two vectors $\bar{a}$ and $\bar{b}$ are not collinear and $m\bar{a} + n\bar{b} = p\bar{a} + q\bar{b}$, then $m = p$, $n = q$.

For example, If two vectors $\bar{a}$ and $\bar{b}$ are not collinear and $3\bar{a} + y\bar{b} = x\bar{a} + 5\bar{b}$, then $3 = x$, $y = 5$.

5.1.6 Coplanar Vectors :

Two or more vectors are coplanar, if they lie in the same plane or in parallel plane.

Vector $\bar{a}$ and $\bar{b}$ are coplanar.

Vector $\bar{a}$ and $\bar{c}$ are coplanar.

Are vectors $\bar{a}$ and $\bar{e}$ coplanar ?

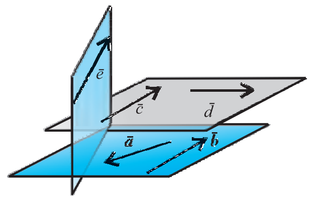


Fig 5.17

Remark : Any two intersecting straight lines OA and OB in space determine a plane. We may choose for convenience the coordinate axes of the plane so that O is origin and axis OX is along one of OA or OB.

Theorem 2 : Let $\bar{a}$ and $\bar{b}$ be non-collinear vectors. A vector $\bar{r}$ is coplanar with $\bar{a}$ and $\bar{b}$ if and only if there exist unique scalars t_1, t_2 such that $\bar{r} = t_1\bar{a} + t_2\bar{b}$.

Proof : Only If-part :

Suppose $\bar{r}$ is coplanar with $\bar{a}$ and $\bar{b}$. To show that their exist unique scalars t_1 and t_2 such that $\bar{r} = t_1\bar{a} + t_2\bar{b}$.

Let $\bar{a}$ be along OA and $\bar{b}$ be along OB. Given a vector $\bar{r}$, with initial point O, Let $\overline{OP} = \bar{r}$, draw lines parallel to OB, meeting OA in M and parallel to OA, meeting OB in N.

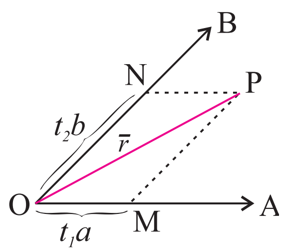


Fig 5.18

Then ON = $t_2\bar{b}$ and OM = $t_1\bar{a}$ for some $t_1, t_2 \in$ R. By triangle law or parallelogram law, we have $\bar{r} = t_1\bar{a} + t_2\bar{b}$

If Part : Suppose $\bar{r} = t_1\bar{a} + t_2\bar{b}$, and we have to show that $\bar{r}$, $\bar{a}$ and $\bar{b}$ are co-planar.

As $\bar{a}$, $\bar{b}$ are coplanar, $t_1\bar{a}$, $t_2\bar{b}$ are also coplanar. Therefore $t_1\bar{a} + t_2\bar{b}$, $\bar{a}$, $\bar{b}$ are coplanar.

Therefore $\bar{a}$, $\bar{b}$, $\bar{r}$ are coplanar.

Uniqueness :

Suppose vector $\bar{r} = t_1\bar{a} + t_2\bar{b}$ ... (1)

can also be written as $\bar{r} = s_1\bar{a} + s_2\bar{b}$ ... (2)

Subtracting (2) from (1) we get,

$\bar{0} = (t_1 - s_1)\bar{a} + (t_2 - s_2)\bar{b}$

But, $\bar{a}$ and $\bar{b}$ are non-collinear, vectors

By Corollary 1 of Theorem 1,

$\therefore t_1 - s_1 = 0 = t_2 - s_2$

$\therefore t_1 = s_1$ and $t_2 = s_2$.

Therefore, the uniqueness follows.

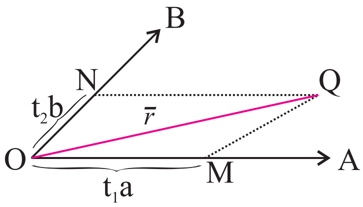


Fig 5.19

Remark :

Linear combination of vectors : If $\bar{a}_1, \bar{a}_2, \bar{a}_3, ..., \bar{a}_n$ are n vectors and $m_1, m_2, m_3,....., m_n$ are n scalars, then the vector $m_1\bar{a}_1 + m_2\bar{a}_2 + m_3\bar{a}_3 + ... + m_n\bar{a}_n$ is called a linear combination of vectors $\bar{a}_1, \bar{a}_2, \bar{a}_3,....., \bar{a}_n$. If atleast one m_i is not zero then the linear combination is non zero linear combination

For example : Let $\bar{a}$, $\bar{b}$ be vectors and *m, n are* scalars then the vector $\bar{c} = m\bar{a} + n\bar{b}$ is called a linear combination of vector $\bar{a}$ and $\bar{b}$. Vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar vectors.

Theorem 3 : Three vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar, if and only if there exists a non-zero linear combination $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ with $(x, y, z) \neq (0, 0, 0)$.

Proof : Only If - part :

Assume that $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar.

Case - 1 : Suppose that any two of $\bar{a}$, $\bar{b}$ and $\bar{c}$ are collinear vectors, say $\bar{a}$ and $\bar{b}$.

$\therefore$ There exist scalars x, y at least one of which is non-zero such that $x\bar{a} + y\bar{b} = \bar{0}$

i.e. $x\bar{a} + y\bar{b} + 0\bar{c} = \bar{0}$ and $(x, y, 0)$ is the required solution for $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$.

Case - 2 : No two vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are collinear.

As $\bar{c}$ is coplanar with $\bar{a}$ and $\bar{b}$,

$\therefore$ we have scalars x, y such that $\bar{c} = x\bar{a} + y\bar{b}$ (using Theorem 2).

$\therefore x\bar{a} + y\bar{b} - \bar{c} = \bar{0}$ and $(x, y, -1)$ is the required solution for $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$.

If - part : Conversely, suppose $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ where one of x, y, z is non-zero, say $z \neq 0$.

$$\therefore \bar{c} = \frac{-x}{z}\bar{a} - \frac{y}{z}\bar{b}$$

$\therefore \bar{c}$ is coplanar with $\bar{a}$ and $\bar{b}$.

$\therefore \bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar vectors.

Corollary 1 : If three vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are not coplanar and $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$, then $x = 0$, $y = 0$ and $z = 0$ because if $(x, y, z) \neq (0, 0, 0)$ then $\bar{a}$, $\bar{b}$, $\bar{c}$ are coplanar.

Corollary 2 : The vectors $\bar{a}$, $\bar{b}$ and $x\bar{a} + y\bar{b}$ are coplanar for all values of x and y.

5.1.7 Vector in Two Dimensions (2-D) :

The plane spanned (covered) by non collinear vectors $\bar{a}$ and $\bar{b}$ is $\{x\bar{a} + y\bar{b} \mid x, y \in R\}$, where $\bar{a}$ and $\bar{b}$ have same initial point.

This is 2-D space where generators are $\bar{a}$ and $\bar{b}$ or its basis is $\{\bar{a}, \bar{b}\}$

For example, in XY plane, let M = (1, 0) and N = (0, 1) be two points along X and Y axis respectively.

Then, we define unit vectors $\hat{i}$ and $\hat{j}$ as $\overline{OM} = \hat{i}$, $\overline{ON} = \hat{j}$.

Given any other vector say $\overline{OP}$, where P = (3, 4) then

$\overline{OP} = 3\hat{i} + 4\hat{j}$

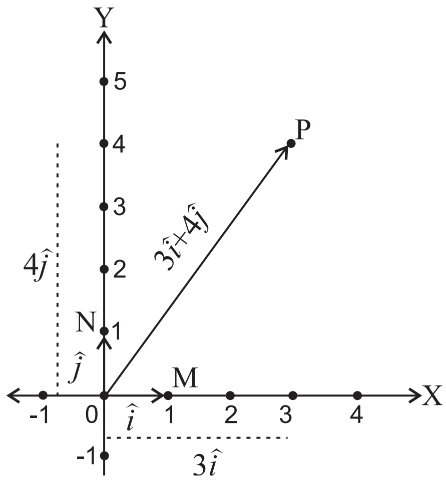


Fig 5.20

5.1.8 Three Dimensional (3-D) Coordinate System :

Any point in the plane is represented as an ordered pair (*a*, *b*) where *a* and *b* are distances (with suitable sign) of point (*a*, *b*) from Y-axis and X-axis respectively.

To locate a point in space, three numbers are required. Here, we need three coordinate axes OX, OY and OZ and to determine a point we need distances of it from three planes formed by these axes.

We represent any point in space by an ordered triple (*a*, *b*, *c*) of real numbers.

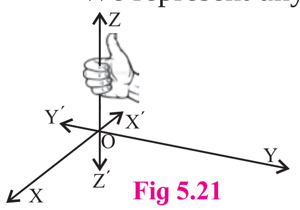


Fig 5.21

O is the origin and three directed lines through O that are perpendicular to each other are the coordinate axes.

Label them as X-axis (XOX'), Y-axis (YOY') and Z-axis (ZOZ'). The direction of Z-axis is determined by right hand rule *i.e.* When you hold your right hand so that the fingers curl from the positive X-axis toward the positive Y-axis, your thumb points along the positive Z-axis, as shown in figure.

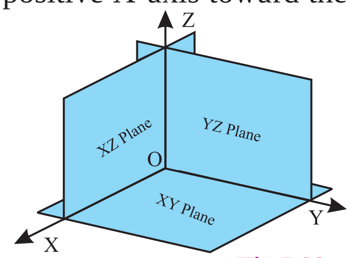


Fig 5.22

Co-ordinates of a point in space :

Let P be a point in the space. Draw perpendiculars PL, PM, PN through P to XY-plane, YZ-plane and XZ-plane respectively, where points L, M and N are feet of perpendicularls in XY, YZ and XZ planes respectively.

For point P(*x*, *y*, *z*), *x*, *y* and *z* are *x*-coordinate, *y*-coordinate and *z*-coordinate respectively.

Point of intersection of all 3 planes is origin O(0, 0, 0).

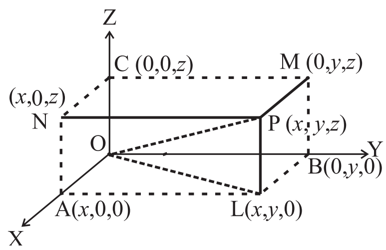


Fig 5.23

Co-ordinates of points on co-ordinate axes :

Points on X-axis, Y-axis and Z-axis have coordinates given by A(*x*, 0, 0), B(0, *y*, 0) and C(0, 0, *z*).

Co-ordinates of points on co-ordinate planes :

Points in XY-plane, YZ-plane and ZX-plane are given by L(*x*, *y*, 0), M(0, *y*, *z*), N(*x*, 0, *z*) respectively.

Distance of P(x, y, z) from co-ordinate planes :

i) Distance of P from XY plane = |PL| = $|z|$.

ii) Distance of P from YZ plane = |PM| = $|x|$.

iii) Distance of P from XZ plane = |PN| = $|y|$.

Distance of any point from origin :

Distance of P (x, y, z) from the origin O(0, 0, 0) from figure 5.23 we have,

$$l(\text{OP}) = \sqrt{\text{OL}^2 + \text{LP}^2} \quad (\Delta\text{OLP right angled triangle})$$

$$= \sqrt{\text{OA}^2 + \text{AL}^2 + \text{LP}^2}$$

$$= \sqrt{\text{OA}^2 + \text{OB}^2 + \text{OC}^2}$$

$$= \sqrt{x^2 + y^2 + z^2}$$

Distance between any two points in space

Distance between two points A(x_1, y_1, z_1) and B(x_2, y_2, z_2) in space is given by distance formula

$$l(\text{AB}) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Distance of a point P(x, y, z) from coordinate axes.

In fig. 5.23, PA is perpendicular to X-axis. Hence distance of P from X-axis is PA.

$$\therefore \ \text{PA} = \sqrt{(x - x)^2 + (y - 0)^2 + (z - 0)^2}$$

$$= \sqrt{y^2 + z^2}$$

iv) In a right-handed system. Octants II, III and IV are found by rotating anti-clockwise around the positive Z-axis. Octant V is vertically below Octant I. Octants VI, VII and VIII are then found by rotating anti-clockwise around the negative Z-axis.

Signs of coordinates of a point P(x, y, z) in different octants :

Octant (x, y, z)	(I) O-XYZ (+, +, +)	(II) O-X'YZ (–, +, +)	(III) O-XY'Z (+, –, +)	(IV) O-X'Y'Z (–, –, +)
Octant (x, y, z)	(V) O-XYZ' (+, +, –)	(VI) O-X'YZ' (–, +, –)	(VII) O-XY'Z' (+, –, –)	(VIII) O-X'Y'Z' (–, –, –)

Point in Octants

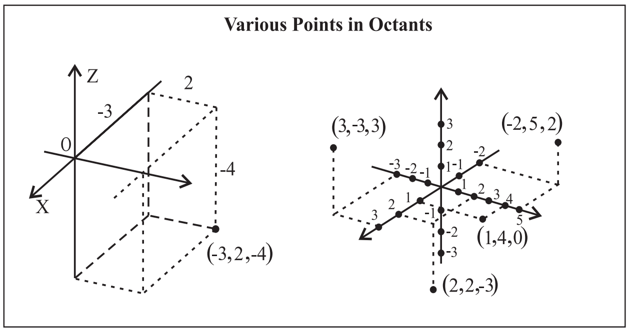


Fig 5.24

Various shapes inb space

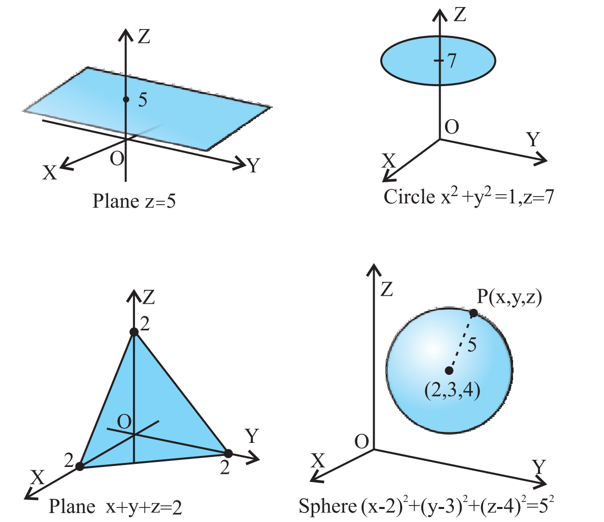


Fig 5.25

5.1.9 Components of Vector :

In order to be more precise about the direction of a vector we can represent a vector as a linear combination of basis vectors.

Take the points A(1, 0, 0), B(0, 1, 0) and C(0, 0, 1) on the X-axis, Y-axis and Z-axis, respectively.

Then $|\overline{OA}| = |\overline{OB}| = |\overline{OC}| = 1$

The vectors $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ each having magnitude 1 are called unit vectors along the axes X, Y, Z respectively. These vectors are denoted by $\hat{i}$, $\hat{j}$, $\hat{k}$ respectively and also called as standard basis vectors or standard unit vectors.

Any vector, along X-axis is a scalar multiple of unit vector $\hat{i}$, along Y-axis is a scalar multiple of $\hat{j}$ and along Z-axis is a scalar multiple of $\hat{k}$. (Collinearity property).

e.g.i) $3\hat{i}$ is a vector along OX with magnitude 3.

ii) $5\hat{j}$ is a vector along OY with magnitude 5.

iii) $4\hat{k}$ is a vector along OZ with magnitude 4.

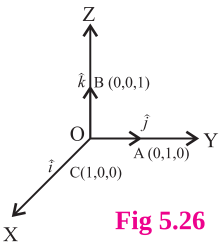


Fig 5.26

Theorem 4 :

If $\bar{a}$, $\bar{b}$, $\bar{c}$ are three non-coplanar vectors, then any vector $\bar{r}$ in the space can be uniquely expressed as a linear combination of $\bar{a}$, $\bar{b}$, $\bar{c}$.

Proof : Let A be any point in the space, take the vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{r}$, so that A becomes their initial point (Fig.5.27).

Let $\overline{AP} = \bar{r}$. As $\bar{a}$, $\bar{b}$, $\bar{c}$ are non-coplanar vectors, they determine three distinct planes intersecting at the point A. Through the point P, draw the plane parallel to the plane formed by vectors $\bar{b}$, $\bar{c}$.

This plane intersects line containing $\bar{a}$ at point B. Similarly, draw the other planes and complete the parallelopiped.

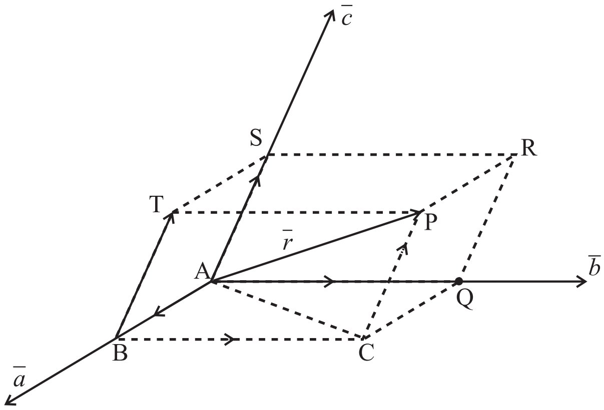


Fig 5.27

Now, $\overline{AB}$ and $\bar{a}$ are collinear.

$\therefore$ $\overline{AB} = x\bar{a}$, where x is a scalar.

Similarly, we have $\overline{AQ} = y\bar{b}$ and $\overline{AS} = z\bar{c}$, where y and z are scalars.

Also, by triangle law of addition of vectors,

$$\overline{AP} = \overline{AC} + \overline{CP} \quad \text{(In } \Delta\text{ACP)}$$

$$= \left(\overline{AB} + \overline{BC}\right) + \overline{CP} \quad \text{(In } \Delta \text{ ABC)}$$

$$= \overline{AB} + \overline{BC} + \overline{CP}$$

$$\overline{AP} = \overline{AB} + \overline{BC} + \overline{CP}$$

$$\therefore \ \bar{r} = x\bar{a} + y\bar{b} + z\bar{c}. \qquad (\because \overline{AQ} = \overline{BC} \text{ and } \overline{AS} = \overline{CP})$$

Therefore, any vector $\bar{r}$ in the space can be expressed as a linear combination of $\bar{a}$, $\bar{b}$ and $\bar{c}$.

Uniqueness :

Suppose $\bar{r} = x_1\bar{a} + x_2\bar{b} + x_3\bar{c}$ and also $\bar{r} = y_1\bar{a} + y_2\bar{b} + y_3\bar{c}$ for some scalars x_1, x_2, x_3 and y_1, y_2, y_3.

We need to prove $x_1 = y_1$, $x_2 = y_2$ and $x_3 = y_3$.

Subtracting one expression from the other we have $(x_1 - y_1)\bar{a} + (x_2 - y_2)\bar{b} + (x_3 - y_3)\bar{c} = \bar{0}$.

By Corollary 1 of Theorem 3

As $\bar{a}$, $\bar{b}$, $\bar{c}$ are non-coplanar we must have $x_1 - y_1 = x_2 - y_2 = x_3 - y_3 = 0$ that is $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, as desired.

5.1.10 Position vector of a point P(x, y, z) in space :

Consider a point P in space, having coordinates (x, y, z) with respect to the origin O(0, 0, 0). Then the vector $\overline{OP}$ having O and P as its initial and terminal points, respectively is called the position vector of the point P with respect to O.

Let P (x, y, z) be a point in space.

$\therefore$ OA = PM = x, OB = PN = y, OC = PL = z.

i.e. A $\equiv (x, 0, 0)$, B $\equiv (0, y, 0)$ and C $\equiv (0, 0, z)$.

Let $\hat{i}$, $\hat{j}$, $\hat{k}$ be unit vectors along positive directions of X-axis, Y-axis and Z-axis is respectively.

$\therefore$ $\overline{OA} = x\hat{i}$, $\overline{OB} = y\hat{j}$ and $\overline{OC} = z\hat{k}$

$\overline{OP}$ is the position vector of point P in space with respect to origin O.

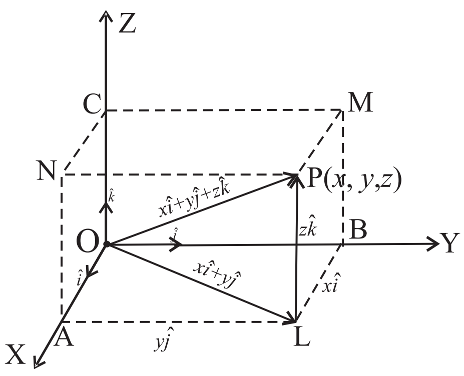


Fig 5.28

Representation of $\overline{OP}$ in terms of unit vector $\hat{i}$, $\hat{j}$, $\hat{k}$.

In ΔOLP we have (see fig. 5.28)

$$\overline{OP} = \overline{OL} + \overline{LP}$$

$$= \overline{OA} + \overline{AL} + \overline{LP} \qquad \text{(from } \Delta\text{OAL)}$$

$$= \overline{OA} + \overline{OB} + \overline{OC} \qquad (\because \overline{AL} = \overline{OB})$$

$$= x\hat{i} + y\hat{j} + z\hat{k} \qquad \dots \dots \dots (1)$$

Magnitude of $\overline{OP}$

Now, $OP^2 = OL^2 + LP^2$ (In right angled ΔOLP)

$= OA^2 + AL^2 + LP^2$ (In right angled ΔOAL)

$= OA^2 + OB^2 + OC^2$

$= x^2 + y^2 + z^2$. ($\because$ AL = OB)

$\therefore\ l(OP) = \sqrt{x^2+y^2+z^2}$

$\therefore\ \left(\overline{OP}\right) = \sqrt{x^2+y^2+z^2}$

5.1.11 Component form of $\bar{r}$:

If $\bar{r}$ is a position vector (p.v.) of point P w.r.t. O then $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

In this representation x, y, z are called the components of $\bar{r}$ along OX, OY and OZ.

Note that any vector in space is unique linear combination of $\hat{i}$, $\hat{j}$ and $\hat{k}$.

Note : Some authors represent vector $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ using angle brackets as $\bar{r} = \langle x, y, z\rangle$ for the point (x, y, z).

5.1.12 Vector joining two points :

Let O be the origin then $\overline{OA}$ and $\overline{OB}$ are the position vectors of points A and B w.r.t. origin 'O'.

In ΔAOB, we have by triangle law.

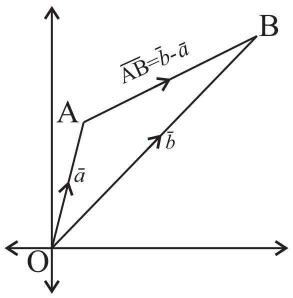


Fig 5.29

$\overline{AB} = \overline{AO} + \overline{OB}$

$= -\overline{OA} + \overline{OB}$ ($\because \overline{AO} = -\overline{OA}$)

$= \overline{OB} - \overline{OA}$

= position vector of B – position vector of A

That is, $\overline{AB} = \bar{b} - \bar{a}$

If A $\equiv (x_1, y_1, z_1)$ and B $\equiv (x_2, y_2, z_2)$

Then $\overline{OA} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\overline{OB} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$\therefore \overline{AB} = \overline{OB} - \overline{OA}$

$$= \left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right)$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\left|\overline{AB}\right| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

In general, any non zero vector $\bar{a}$ in space can be expressed uniquely as the linear combination of $\hat{i}$, $\hat{j}$, $\hat{k}$ as $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ where a_1, a_2, a_3 are scalars.

$\therefore |\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Note :

(i) If $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\bar{a} = \bar{b}$ if $a_1 = b_1, a_2 = b_2, a_3 = b_3$.

(ii) If $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\bar{a} + \bar{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) + \left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}.$$

(iii) If k is any scalar then $k\bar{a} = k\left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) = ka_1\hat{i} + ka_2\hat{j} + ka_3\hat{k}$

Also if $\bar{b}$ and $\bar{a}$ are collinear i.e. $\bar{b} = k\bar{a}$ then $\dfrac{b_1}{a_1} = \dfrac{b_2}{a_2} = \dfrac{b_3}{a_3} = k$.

(iv) Let $\hat{a}$ be a unit vector along a non zero vector $\bar{a}$ in space, Then $\hat{a} = \dfrac{\bar{a}}{|\bar{a}|} = \dfrac{a_1\hat{i} + a_2\hat{j} + a_3\hat{k}}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$.

(v) **(Collinearity of 3-points)** Three distinct points A, B and C with position vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ respectively are collinear if and only if there exist three non-zero scalars x, y and z such that $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ and $x + y + z = 0$. (Use Theorem 1 and the fact that $\overline{AB}$ and $\overline{BC}$ are collinear).

(vi) **(Coplanarity of 4-points)** Four distinct points A, B, C and D (no three of which are collinear) with position vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ respectively are coplanar if and only if there exist four scalars x, y, z and w, not all zero, such that $x\bar{a} + y\bar{b} + z\bar{c} + w\bar{d} = \bar{0}$,

where $x + y + z + w = 0$. (Use Theorem 2 and the fact that $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$ are coplanar).

(vii) **Linearly dependent vectors :** A set of non-zero vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ is said to be linearly dependent if there exist scalars x, y, z not all zero such that $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$.

Such vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar.

(viii) **Linearly independent vectors :** A set of non-zero vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ is said to be linearly independent if $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$, then $x = y = z = 0$.

Such vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are non-coplanar.

Solved Examples

Ex.1. State the vectors which are :

(i) equal in magnitude

(ii) parallel

(iii) in the same direction

(iv) equal

(v) negatives of one another

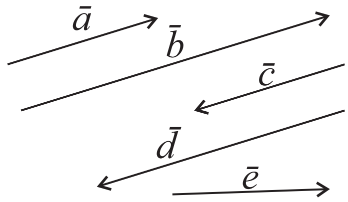


Fig 5.30

Solution :

(i) $\bar{a}$, $\bar{c}$ and $\bar{e}$; $\bar{b}$ and $\bar{d}$

(ii) $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$

(iii) $\bar{a}$ and $\bar{b}$; $\bar{c}$ and $\bar{d}$

(iv) none are equal

(v) $\bar{a}$ and $\bar{c}$, $\bar{b}$ and $\bar{d}$

Ex. 2. In the diagram $\overrightarrow{KL} = \bar{a}$, $\overrightarrow{LN} = \bar{b}$, $\overrightarrow{NM} = \bar{c}$ and $\overrightarrow{KT} = \bar{d}$. Find in terms of $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$. (i) $\overrightarrow{LT}$ (ii) $\overrightarrow{KM}$ (iii) $\overrightarrow{TN}$ (iv) $\overrightarrow{MT}$

Solution :

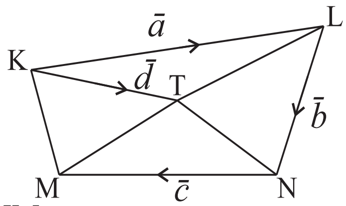


Fig 5.31

(i) In ΔKLT, using triangle law

$\overline{KL} + \overline{LT} = \overline{KT}$ i.e. $\bar{a} + \overline{LT} = \bar{d}$

$\overline{LT} = \bar{d} - \bar{a}$

(ii) Using polygonal law of addition of vectors for polygon KLNM

$\overline{KM} = \overline{KL} + \overline{LN} + \overline{NM}$.

$= \bar{a} + \bar{b} + \bar{c}$

(iii) Using polygonal law of addition of vectors for polygon TKLN

$\overline{TN} = \overline{TK} + \overline{KL} + \overline{LN}$

$= -\bar{d} + \bar{a} + \bar{b} = \bar{a} + \bar{b} - \bar{d}$

(iv) Using polygonal law of addition of vectors for polygon TKLNM

$\overline{MT} = \overline{MN} + \overline{NL} + \overline{LK} + \overline{KT}$

$= -\bar{c} - \bar{b} - \bar{a} + \bar{d}$

$= \bar{d} - \bar{a} - \bar{b} - \bar{c}$.

Ex.3. Find the magnitude of following vectors :

(i) $\bar{a} = \hat{i} - 2\hat{j} + 4\hat{k}$ (ii) $\bar{b} = 4\hat{i} - 3\hat{j} - 7\hat{k}$

(iii) a vector with initial point : (1, – 3, 4); terminal point : (1, 0, – 1).

Solution :

(i) $|\bar{a}| = \sqrt{1^2 + (-2)^2 + 4^2} = \sqrt{21}$

(ii) $|\bar{b}| = \sqrt{4^2 + (-3)^2 + (-7)^2}$

$|\bar{b}| = \sqrt{16 + 9 + 49} = \sqrt{74}$

(iii) $|\bar{c}| = \left(\hat{i}-\hat{k}\right)-\left(\hat{i}-3\hat{j}+4\hat{k}\right) = 3\hat{j}-5\hat{k}$

$|\bar{c}| = \sqrt{0+3^2+(-5)^2} = \sqrt{34}$

Ex. 4. A(2, 3), B (– 1, 5), C (– 1, 1) and D (– 7, 5) are four points in the Cartesian plane.

(i) Find $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

(ii) Check if, $\overrightarrow{CD}$ is parallel to $\overrightarrow{AB}$.

(iii) E is the point (*k*, l) and $\overrightarrow{AC}$ is parallel to $\overrightarrow{BE}$. Find *k*.

Solution : (i) $\bar{a} = 2\hat{i}+3\hat{j}, \bar{b} = -\hat{i}+5\hat{j}, \bar{c} = -\hat{i}+\hat{j}, \bar{d} = -7\hat{i}+5\hat{j}$

$\overline{AB} = \bar{b}-\bar{a} = \left(-\hat{i}+5\hat{j}\right)-\left(2\hat{i}+3\hat{j}\right) = -3\hat{i}+2\hat{j}$

$\overrightarrow{CD} = \bar{d}-\bar{c} = \left(-7\hat{i}+5\hat{j}\right)-\left(-\hat{i}+\hat{j}\right) = -6\hat{i}+4\hat{j}$

(ii) $\overline{CD} = -6\hat{i}+4\hat{j} = 2\left(-3\hat{i}+2\hat{j}\right) = 2\ \overline{AB}$ therefore $\overrightarrow{CD}$ and $\overrightarrow{AB}$ are parallel.

(iii) $\overrightarrow{BE} = \left(k\hat{i}+\hat{j}\right)-\left(-\hat{i}+5\hat{j}\right) = (k+1)\hat{i}-4\hat{j}$

$\overrightarrow{AC} = \left(-\hat{i}+\hat{j}\right)-\left(2\hat{i}+3\hat{j}\right) = -3\hat{i}-2\hat{j}$

$\overrightarrow{BE} = m\overrightarrow{AC}$

$(k+1)\hat{i}-4\hat{j} = m\left(-3\hat{i}-2\hat{j}\right)$

So – 4 therefore 2 = m

and $k+1 = -3m$

$k+1 = -3(2)$

$k = -6-1$

$k = -7$

Ex. 5. Determine the values of *c* that satisfy $|c\bar{u}| = 3, \bar{u} = \hat{i}+2\hat{j}+3\hat{k}$

Solution : $|c\bar{u}| = \sqrt{c^2+4c^2+9c^2} = |c|\sqrt{14} = 3$

$\therefore c = \pm\frac{3}{\sqrt{14}}$

Ex. 6. Find a unit vector (i) in the direction of $\bar{u}$ and (ii) in the direction opposite of $\bar{u}$. where $\bar{u} = 8\hat{i}+3\hat{j}-\hat{k}$.

Solution : (i) $\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{8\hat{i}+3\hat{j}-1\hat{k}}{\sqrt{74}}$

$= \frac{1}{\sqrt{74}}\left(8\hat{i}+3\hat{j}-\hat{k}\right)$ is the unit vector in direction of $\bar{u}$.

(ii) $-\hat{u} = -\frac{1}{\sqrt{74}}\left(8\hat{i}+3\hat{j}-\hat{k}\right)$ is the unit vector in opposite direction of $\bar{u}$.

Ex. 7. Show that the vectors $2\hat{i}-3\hat{j}+4\hat{k}$ and $-4\hat{i}+6\hat{j}-8\hat{k}$ are parallel.

Solution : $\bar{a}=2\hat{i}-3\hat{j}+4\hat{k}$

$\bar{b}=-4\hat{i}+6\hat{j}-8\hat{k}=-2\left(2\hat{i}-3\hat{j}+4\hat{k}\right)=-2\bar{a}$

As $\bar{b}$ is scalar multiple of $\bar{a}$

$\therefore$ $\bar{b}$ and $\bar{a}$ are parallel.

Ex. 8. The non-zero vectors $\bar{a}$ and $\bar{b}$ are not collinear find the value of λ and μ :

(i) $\bar{a}+3\bar{b}=2\lambda\bar{a}-\mu\bar{b}$

(ii) $(1+\lambda)\bar{a}+2\lambda\bar{b}=\mu\bar{a}+4\mu\bar{b}$

(iii) $(3\lambda+5)\bar{a}+\bar{b}=2\mu\bar{a}+(\lambda-3)\bar{b}$

Solution : (i) $2\lambda = 1, 3 = -\mu \therefore \lambda = ½, \mu = -3$

(ii) $1 + \lambda = \mu, 2\lambda = 4\mu, \lambda = 2\mu$

$1 + 2\mu = \mu, 1 = -\mu$

$\therefore \mu = -1,\ \lambda = -2$

(iii) $3\lambda + 5 = 2\mu, 1 = \lambda - 3, \therefore \lambda = 1 + 3 = 4$

and $3(4) + 5 = 2\mu$

$\therefore 2\mu = 17$. So $\mu = \dfrac{17}{2}$

Ex. 9. Are the following set of vectors linearly independent?

(i) $\bar{a}=\hat{i}-2\hat{j}+3\hat{k},\ \bar{b}=3\hat{i}-6\hat{j}+9\hat{k}$

(ii) $\bar{a}=-2\hat{i}-4\hat{k},\quad \bar{b}=\hat{i}-2\hat{j}-\hat{k},\quad \bar{c}=\hat{i}-4\hat{j}+3\hat{k}$. Interpret the results.

Solution : (i) $\bar{a}=\hat{i}-2\hat{j}+3\hat{k},\bar{b}=3\hat{i}-6\hat{j}+9\hat{k}$

$\bar{b}=3\left(\hat{i}-2\hat{j}+3\hat{k}\right)$

$\bar{b}=3\bar{a}$ Here $\bar{a}$ and $\bar{b}$ linearly dependent vectors. Hence $\bar{a}$ and $\bar{b}$ are collinear.

(ii) $\bar{a}=-2\hat{i}-4\hat{k}$, $\bar{b}=\hat{i}-2\hat{j}-\hat{k}$, $\bar{c}=\hat{i}-4\hat{j}+3\hat{k}$

Let $x\bar{a}+y\bar{b}+z\bar{c}=\bar{0}$

$\therefore x\left(-2\hat{i}-4\hat{k}\right)+y\left(\hat{i}-2\hat{j}-\hat{k}\right)+z\left(\hat{i}-4\hat{j}+3\hat{k}\right)=\bar{0}$

$-2x+y+z=0$

$-2y-4z=0$

$-4x-y+3z=0$

$\therefore x=0, y=0, z=0$. Here $\bar{a}$, $\bar{b}$ and $\bar{c}$ are linearly independent vectors.

Hence $\bar{a}$, $\bar{b}$ and $\bar{c}$ are non-coplanar.

Ex. 10. If $\bar{a} = 4\hat{i}+3\hat{k}$ and $\bar{b} = -2\hat{i}+\hat{j}+5\hat{k}$ find (i) $|\bar{a}|$, (ii) $\bar{a} + \bar{b}$, (iii) $\bar{a} - \bar{b}$, (iv) $3b$, (v) $2\bar{a} + 5\bar{b}$

Solution : (i) $|\bar{a}| = \sqrt{4^2+0^2+3^2} = \sqrt{25} = 5$

(ii) $\bar{a}+\bar{b} = \left(4\hat{i}+3\hat{k}\right)+\left(-2\hat{i}+\hat{j}+5\hat{k}\right) = 2\hat{i}+\hat{j}+8\hat{k}$

(iii) $\bar{a}-\bar{b} = \left(4\hat{i}+3\hat{k}\right)-\left(-2\hat{i}+\hat{j}+5\hat{k}\right) = 6\hat{i}-\hat{j}-2\hat{k}$

(iv) $3\bar{b} = 3\left(-2\hat{i}+\hat{j}+5\hat{k}\right) = -6\hat{i}+3\hat{j}+15\hat{k}$

(v) $2\bar{a}+5\bar{b} = 2\left(4\hat{i}+3\hat{k}\right)+5\left(-2\hat{i}+\hat{j}+5\hat{k}\right)$

$= \left(8\hat{i}+6\hat{k}\right)+\left(-10\hat{i}+5\hat{j}+25\hat{k}\right) = -2\hat{i}+5\hat{j}+31\hat{k}$

Ex. 11. What is the distance from the point (2, 3, 4) to (i) the XY plane? (ii) the X-axis? (iii) origin (iv) point (-2, 7, 3).

Solution :

(a) The distance from (2, 3, 4) to the XY plane is $|z|$ = 4 units.

(b) The distance from (2, 3, 4) to the X-axis is $\sqrt{y^2+z^2} = \sqrt{3^2+4^2} = \sqrt{25}$ = 5 units.

(c) The distance from $(x, y, z) \equiv (2, 3, 4)$ to origin (0, 0, 0) is

$\sqrt{x^2+y^2+z^2} = \sqrt{2^2+3^2+4^2} = \sqrt{4+9+16} = \sqrt{29}$ units.

(d) The distance from (2, 3, 4) to (-2, 7, 3) is

$\sqrt{(2+2)^2+(3-7)^2+(4-3)^2} = \sqrt{16+16+1} = \sqrt{33}$ units.

Ex. 12. Prove that the line segment joining the midpoints of two sides of a triangle is parallel to and half of the third side.

Solution : Let the triangle be ABC. If M and N are the midpoints of AB and AC respectively, then

$\overrightarrow{AM} = ½\,\overrightarrow{AB}$ and $\overrightarrow{AN} = ½\,\overrightarrow{AC}$. Thus by triangle law

$\overrightarrow{AN} = \overrightarrow{AM} + \overrightarrow{MN}$

$\therefore \overrightarrow{MN} = \overrightarrow{AN} - \overrightarrow{AM} = \dfrac{\overrightarrow{AC}-\overrightarrow{AB}}{2} = \dfrac{\overrightarrow{BC}}{2}$

Thus, $\overrightarrow{MN}$ is parallel to and half as long as $\overrightarrow{BC}$.

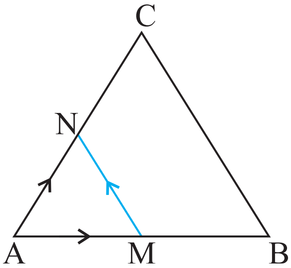


Fig 5.32

Ex. 13. In quadrilateral ABCD, M and N are the mid-points of the diagonals AC and BD respectively. Prove that $\overline{AB} + \overline{AD} + \overline{CB} + \overline{CD} = 4\overline{MN}$

Solution : $\overline{AB} = \overline{AM} + \overline{MN} + \overline{NB}$ ... (1)

$\overline{AD} = \overline{AM} + \overline{MN} + \overline{ND}$ ... (2)

$\overline{CB} = \overline{CM} + \overline{MN} + \overline{NB}$ ... (3)

$\overline{CD} = \overline{CM} + \overline{MN} + \overline{ND}$ ... (4)

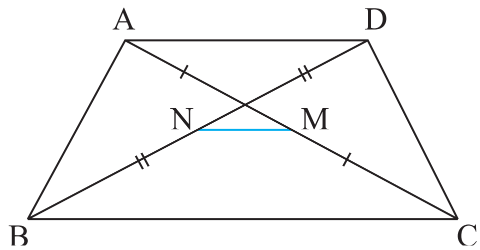


Fig 5.33

Add (1), (2), (3), (4) to get

$\overline{AB} + \overline{AD} + \overline{CB} + \overline{CD} = 2\overline{AM} + 2\overline{CM} + 4\overline{MN} + 2\overline{NB} + 2\overline{ND}$

$= 2(\overline{AM} + \overline{CM}) + 4\overline{MN} + 2(\overline{NB} + \overline{ND})$

$= 2(\overline{AM} - \overline{AM}) + 4\overline{MN} + 2(\overline{NB} - \overline{NB})$ ($\because$ $\overline{MC} = \overline{AM}$ and $\overline{DN} = \overline{NB}$)

$= 2\times(\overline{0}) + 4\overline{MN} + 2\times(\overline{0})$

$= 4\overline{MN}$

Ex. 14. Express $-\hat{i} - 3\hat{j} + 4\hat{k}$ as the linear combination of the vectors $2\hat{i} + \hat{j} - 4\hat{k}$, $2\hat{i} - \hat{j} + 3\hat{k}$ and $3\hat{i} + \hat{j} - 2\hat{k}$.

Solution : Let $\overline{r} = -\hat{i} - 3\hat{j} + 4\hat{k}$ $\overline{a} = 2\hat{i} + \hat{j} - 4\hat{k}$, $\overline{b} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\overline{c} = 3\hat{i} + \hat{j} - 2\hat{k}$

Consider $\overline{r} = x\overline{a} + y\overline{b} + z\overline{c}$

$-\hat{i} - 3\hat{j} + 4\hat{k} = x\left(2\hat{i} + \hat{j} - 4\hat{k}\right)\hat{i} + \left(2\hat{i} - \hat{j} + 3\hat{k}\right) + z\left(3\hat{i} + \hat{j} - 2\hat{k}\right)$

$-\hat{i} - 3\hat{j} + 4\hat{k} = (2x + 2y + 3z)\hat{i} + (x - y + z)\hat{j} + (-4x + 3y - 2z)\hat{k}$

By equality of vectors, we get $-1 = 2x + 2y + 3z$, $-3 = x - y + z$, $4 = -4x + 3y - 2z$,

Using Cramer's rule we get, $x = 2$, $y = 2$, $z = 3$. Therefore $\overline{r} = 2\overline{a} + 2\overline{b} - 3\overline{c}$

Ex. 15. Show that the three points A(1, –2, 3), B(2, 3, –4) and C(0, –7, 10) are collinear.

Solution : If $\overline{a}$, $\overline{b}$ and $\overline{c}$ are the position vectors of the points A, B and C respectively, then

$\overline{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

$\overline{b} = 2\hat{i} + 3\hat{j} - 4\hat{k}$

$\overline{c} = 0\hat{i} - 7\hat{j} + 10\hat{k}$

$\overline{AB} = \overline{b} - \overline{a} = \hat{i} + 5\hat{j} - 7\hat{k}$ ...(1) and

$\overline{AC} = \overline{c} - \overline{a}$

$= -\hat{i} - 5\hat{j} + 7\hat{k}$

$= (-1)\left[\hat{i} + 5\hat{j} - 7\hat{k}\right]$

$\overline{AC} = (-1)\overline{AB}$ ...from (1)

That is, $\overline{AC}$ is a scalar multiple of $\overline{AB}$. Therefore, they are parallel. But point A is in common. Hence, the points A, B and C are collinear.

Ex. 16. Show that the vectors $4\hat{i}+13\hat{j}-18\hat{k}$, $\hat{i}-2\hat{j}+3\hat{k}$ and $2\hat{i}+3\hat{j}-4\hat{k}$ are coplanar.

Solution : Let , $\bar{a}=4\hat{i}+13\hat{j}-18\hat{k}$, $\bar{b}=\hat{i}-2\hat{j}+3\hat{k}$, $\bar{c}=2\hat{i}+3\hat{j}-4\hat{k}$

Consider $\bar{a}=m\bar{b}+n\bar{c}$

$$\therefore \quad 4\hat{i}+13\hat{j}-18\hat{k}=m\left(\hat{i}-2\hat{j}+3\hat{k}\right)+n\left(2\hat{i}+3\hat{j}-4\hat{k}\right)$$

$$\therefore \quad 4\hat{i}+13\hat{j}-18\hat{k}=(m+2n)\hat{i}+(-2m+3n)\hat{j}+(3m-4n)\hat{k}$$

By equality of two vectors, we have

$$m+2n=4 \qquad \ldots(1)$$

$$-2m+3n=13 \qquad \ldots(2)$$

$$3m-4n=-18 \qquad \ldots(3)$$

Solving (1) and (2) we get, $\therefore m=-2$, $\quad n=3$

These values of *m* and *n* satisfy equation (3) also.

$$\therefore \bar{a}=-2\bar{b}+3\bar{c}$$

Therefore, $\bar{a}$ is a linear combination of $\bar{b}$ and $\bar{c}$. Hence, $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar.

Exercise 5.1

1. The vector $\bar{a}$ is directed due north and $|\bar{a}|$ = 24. The vector $\bar{b}$ is directed due west and $|\bar{b}|$ = 7. Find $|\bar{a}+\bar{b}|$.

2. In the triangle PQR, $\overline{PQ}=2\bar{a}$ and $\overline{QR}=2\bar{b}$. The mid-point of PR is M. Find following vectors in terms of $\bar{a}$ and $\bar{b}$.

   (i) $\overline{PR}$ (ii) $\overline{PM}$ (iii) $\overline{QM}$

3. OABCDE is a regular hexagon. The points A and B have position vectors $\bar{a}$ and $\bar{b}$ respectively, referred to the origin O. Find, in terms of $\bar{a}$ and $\bar{b}$ the position vectors of C, D and E.

4. If ABCDEF is a regular hexagon, show that

   $\overline{AB}+\overline{AC}+\overline{AD}+\overline{AE}+\overline{AF}=6\overline{AO}$, where O is the center of the hexagon.

5. Check whether the vectors $2\hat{i}+2\hat{j}+3\hat{k}$, $-3\hat{i}+3\hat{j}+2\hat{k}$ and $3\hat{i}+4\hat{k}$ form a triangle or not.

6. In the figure 5.34 express $\bar{c}$ and $\bar{d}$ in terms of $\bar{a}$ and $\bar{b}$.

   Find a vector in the direction of $\bar{a}=\hat{i}-2\hat{j}$ that has magnitude 7 units.

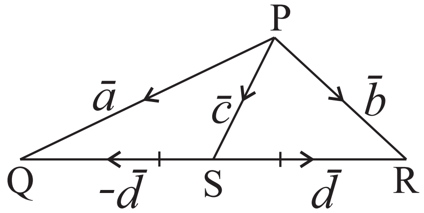


Fig 5.34

7. Find the distance from (4, -2, 6) to each of the following :

   (a) The XY-plane (b) The YZ-plane
   (c) The XZ-plane (d) The X-axis
   (e) The Y-axis (f) The Z-axis

8. Find the coordinates of the point which is located :

   (a) Three units behind the YZ-plane, four units to the right of the XZ-plane and five units above the XY-plane.

   (b) In the YZ-plane, one unit to the right of the XZ-plane and six units above the XY-plane.

9. Find the area of the triangle with vertices (1, 1, 0), (1, 0, 1) and (0, 1, 1).

10. If $\overline{AB} = 2\hat{i} - 4\hat{j} + 7\hat{k}$ and initial point $A \equiv (1,5,0)$. Find the terminal point B.

11. Show that the following points are collinear :

    (i) A (3, 2, -4), B (9, 8, -10), C (-2, -3, 1).

    (ii) P (4, 5, 2), Q (3, 2, 4), R (5, 8, 0).

12. If the vectors $2\hat{i} - q\hat{j} + 3\hat{k}$ and $4\hat{i} - 5\hat{j} + 6\hat{k}$ are collinear, then find the value of q.

13. Are the four points A(1, –1, 1), B(–1, 1, 1), C(1, 1, 1) and D(2, –3, 4) coplanar? Justify your answer.

14. Express $-\hat{i} - 3\hat{j} + 4\hat{k}$ as linear combination of the vectors $2\hat{i} + \hat{j} - 4\hat{k}$, $2\hat{i} - \hat{j} + 3\hat{k}$ and $3\hat{i} + \hat{j} - 2\hat{k}$.

5.2.1 Section Formula :

Theorem 5 : (Section formula for internal division) Let A($\bar{a}$) and B($\bar{b}$) be any two points in the space and R($\bar{r}$) be a point on the line segment AB dividing it internally in the ratio $m : n$.

Then $\bar{r} = \dfrac{m\bar{b} + n\bar{a}}{m+n}$

Proof : As R is a point on the line segment AB (A-R-B) and $\overline{AR}$ and $\overline{RB}$ are in same direction.

$\dfrac{AR}{RB} = \dfrac{m}{n}$, so n (AR) = m (RB)

As $m\left(\overline{RB}\right)$ and $n\left(\overline{AR}\right)$ have same direction and magnitude,

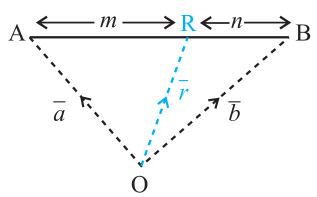


Fig 5.35

$\therefore\ m\left(\overline{RB}\right) = n\left(\overline{AR}\right)$

$\therefore\ m\left(\overline{OB} - \overline{OR}\right) = n\left(\overline{OR} - \overline{OA}\right)$

$\therefore\ m\left(\bar{b} - \bar{r}\right) = n\left(\bar{r} - \bar{a}\right)$

$\therefore\ m\bar{b} - m\bar{r} = n\bar{r} - n\bar{a}$

$\therefore\ m\bar{b} + n\bar{a} = m\bar{r} + n\bar{r} = (m+n)\,\bar{r}$

$\therefore\ \bar{r} = \dfrac{m\bar{b} + n\bar{a}}{m+n}$

Note :

1. If $A \equiv (a_1, a_2, a_3), B \equiv (b_1, b_2, b_3)$ and $R \equiv (r_1, r_2, r_3)$ divides the segment AB in the ratio $m : n$, then $\bar{r}_i = \dfrac{m\bar{b}_i + n\bar{a}_i}{m+n}$, i = 1, 2, 3.

2. **(Midpoint formula)** If R($\bar{r}$) is the mid-point of the line segment AB then m = n so $m : n = 1 : 1$,

$$\therefore \ \bar{r} = \frac{1\bar{b} + 1\bar{a}}{1+1} \text{ that is } \bar{r} = \frac{\bar{a}+\bar{b}}{2}.$$

Theorem 6 : (Section formula for external division) Let A($\bar{a}$) and B($\bar{b}$) be any two points in the space and R($\bar{r}$) be the third point on the line AB dividing the segment AB externally in the ratio $m : n$. Then $\bar{r} = \dfrac{m\bar{b} - n\bar{a}}{m-n}$.

Proof : As the point R divides line segment AB externally, we have either A-B-R or R-A-B.

Assume that A-B-R and $\overline{\text{AR}} : \overline{\text{BR}} = m : n$

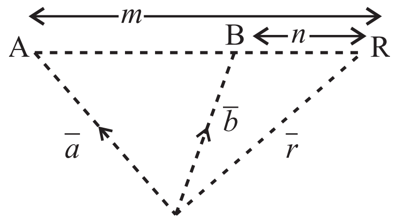


Fig 5.36

$$\therefore \frac{AR}{BR} = \frac{m}{n} \text{ so n(AR) = m (BR)}$$

As $n\left(\overline{\text{AR}}\right)$ and $m\left(\overline{\text{BR}}\right)$ have same magnitude and direction,

$$\therefore n\left(\overline{\text{AR}}\right) = m\left(\overline{\text{BR}}\right)$$

$$\therefore n\left(\bar{r} - \bar{a}\right) = m\left(\bar{r} - \bar{b}\right)$$

$$\therefore n\bar{r} - n\bar{a} = m\bar{r} - m\bar{b}$$

$$\therefore m\bar{b} - n\bar{a} = m\bar{r} - n\bar{r} = (m-n)\bar{r}$$

$$\therefore \bar{r} = \frac{m\bar{b} - n\bar{a}}{m-n}$$

Note : 1) Whenever the ratio in which point R divides the join of two points A and B is required, it is convenient to take the ratio as $k : 1$.

Then, $\bar{r} = \dfrac{k\bar{b} + \bar{a}}{k+1}$, if division is internal,

$\bar{r} = \dfrac{k\bar{b} - \bar{a}}{k-1}$, if division is external.

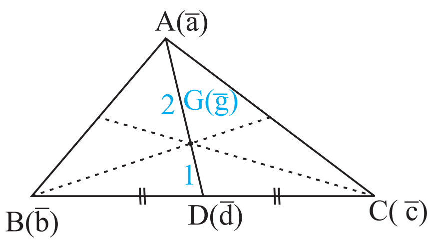


Fig 5.37

2. In ΔABC, centroid G divides the medians internally in ratio 2 : 1 and is given by (see fig. 5.37)

$$\bar{g} = \frac{\bar{a} + \bar{b} + \bar{c}}{3} \text{ (Verify).}$$

3. In tetrahedron ABCD centroid G divides the line joining the vertex of tetrahedron to centroid of opposite triangle in the ratio 3 : 1 and is given by $\bar{g} = \dfrac{\bar{a} + \bar{b} + \bar{c} + \bar{d}}{4}$.

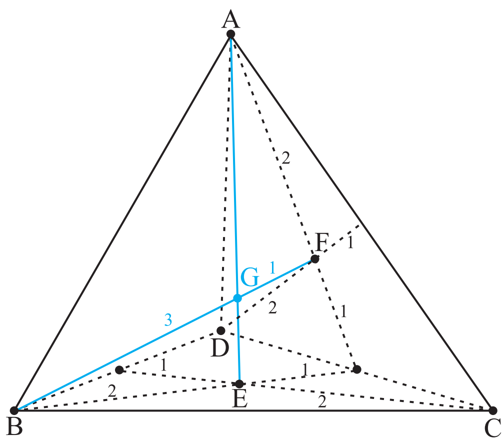


Fig 5.38

Solved Examples

Ex.1. Find the co-ordinates of the point which divides the line segment joining the points A(2, -6, 8)and B(-l, 3, -4). (i) Internally in the ratio 1 : 3. (ii) Externally in the ratio 1: 3.

Solution : If $\bar{a}$ and $\bar{b}$ are position vectors of the points A and B respectively, then

$\bar{a}=2\hat{i}-6\hat{j}+8\hat{k}$ and $\bar{b}=-\hat{i}+3\hat{j}-4\hat{k}$ ·

Suppose R($\bar{r}$) is the point which divides the line segment joining the points A($\bar{a}$) and B($\bar{b}$) internally in the ratio 1 : 3 then,

$$\bar{r}=\frac{1(\bar{b})+3(\bar{a})}{1+3}=\frac{-1(\hat{i}+3\hat{j}-4\hat{k})+3(2\hat{i}-6\hat{j}+8\hat{k})}{4}$$

$$\therefore\ \bar{r}=\frac{5\hat{i}-15\hat{j}+20\hat{k}}{4}$$

∴ The coordinates of the point R are $\left(\frac{5}{4},\frac{-15}{4},5\right)$.

Suppose $S(\bar{s})$ is the point which divides the line joining the points A($\bar{a}$) and B($\bar{b}$) externally in the ratio 1 : 3 then,

$$\bar{s}=\frac{1\bar{b}-3\bar{a}}{1-3}=\frac{(-\hat{i}+3\hat{j}-4\hat{k})-3(2\hat{i}-6\hat{j}+8\hat{k})}{-2}$$

$$\therefore\ \bar{s}=\frac{-7\hat{i}+21\hat{j}-28\hat{k}}{-2}$$

∴ The coordinates of the point S are $\left(\frac{7}{2},\frac{-21}{2},14\right)$.

Ex. 2. If the three points A(3, 2, p), B(q, 8, -10), C(-2, -3, 1) are collinear then find

(i) the ratio in which the point C divides the line segment AB, (ii) the vales of p and q.

Solution : Let $\bar{a}=3\hat{i}+2\hat{j}+p\hat{k}$, $\bar{b}=q\hat{i}+8\hat{j}-10\hat{k}$ and $\bar{c}=-2\hat{i}-3\hat{j}+\hat{k}$.

Suppose the point C divides the line segment AB in the ratio t : 1,

then by section formula, $\bar{c}=\frac{t\bar{b}+1\bar{a}}{t+1}$.

$$\therefore\quad -2\hat{i}-3\hat{j}+\hat{k}=\frac{t(q\hat{i}+8\hat{j}-10\hat{k})+1(3\hat{i}+2\hat{j}+p\hat{k})}{t+1}$$

$$\therefore\quad (-2\hat{i}-3\hat{j}+\hat{k})(t+1)=(tq+3)\hat{i}+(8t+2)\hat{j}+(-10t+p)\hat{k}$$

$$\therefore\quad -2(t+1)\hat{i}-3(t+1)\hat{j}+(t+1)\hat{k}=(tq+3)\hat{i}+(8t+2)\hat{j}+(-10t+p)\hat{k}$$

Using equality of two vectors $-2(t+1)=tq+3$ ... (1)

$-3(t+1)=8t+2$ ... (2)

$t+1=-10t+p$ ... (3)

Now equation (2) gives $t=-\frac{5}{11}$

Put $t=-\frac{5}{11}$ in equation (1) and equation (3), we get q = 9 and p = -4.
The negative sign of t suggests that the point C divides the line segment AB externally in the ratio 5:11.

Ex.3. If A(5, 1, p), B(1, q, p) and C(1, –2, 3) are vertices of triangle and $G\left(r,-\frac{4}{3},\frac{1}{3}\right)$ is its centroid, then find the values of p, q and r.

Solution : Let $\bar{a}=5\hat{i}+\hat{j}+p\hat{k}$, $\quad \bar{b}=\hat{i}+q\hat{j}+p\hat{k}$

$\bar{c}=\hat{i}-2\hat{j}+3\hat{k}$ and $\bar{g}=r\hat{i}-\frac{4}{3}\hat{j}+\frac{1}{3}\hat{k}$

By centroid formula we have $\bar{g}=\frac{\bar{a}+\bar{b}+\bar{c}}{3}$

$$\therefore \quad 3\bar{g}=\bar{a}+\bar{b}+\bar{c}$$

$$\therefore \quad 3\left(r\hat{i}-\frac{4}{3}\hat{j}+\frac{1}{3}\hat{k}\right)=\left(5\hat{i}+\hat{j}+p\hat{k}\right)+\left(\hat{i}+q\hat{j}+p\hat{k}\right)+\left(\hat{i}-2\hat{j}+3\hat{k}\right)$$

$$\therefore \quad (3r)\hat{i}-4\hat{j}+\hat{k}=7\hat{i}+(q-1)\hat{j}+(2p+3)\hat{k}$$

$$\therefore \quad 3r=7,-4=q-1,1=2p+3$$

$$\therefore \quad r=\frac{7}{3},\quad q=-3,\quad p=-1$$

Ex.4. If $\bar{a}$, $\bar{b}$, $\bar{c}$ are the position vectors of the points A, B, C respectively and $5\bar{a}-3\bar{b}-2\bar{c}=\bar{0}$, then find the ratio in which the point C divides the line segment BA.

Solution : As $5\bar{a}-3\bar{b}-2\bar{c}=\bar{0}$

$$\therefore \quad 2\bar{c}=5\bar{a}-3\bar{b}$$

$$\therefore \quad \bar{c}=\frac{5\bar{a}-3\bar{b}}{2}$$

$$\therefore \quad \bar{c}=\frac{5\bar{a}-3\bar{b}}{5-3}$$

$\therefore$ This shows that the point C divides BA externally in the ratio 5 : 3.

Ex.5. Prove that the medians of a triangle are concurrent.

Solution : Let A, B and C be vertices of a triangle. Let D, E and F be the mid-points of the sides BC, AC and AB respectively. Let $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$ and $\bar{f}$ be position vectors of points A, B, C, D, E and F respectively.

Therefore, by mid-point formula,

$$\therefore \bar{d}=\frac{\bar{b}+\bar{c}}{2},\quad \bar{e}=\frac{\bar{a}+\bar{c}}{2} \text{ and } \bar{f}=\frac{\bar{a}+\bar{b}}{2}$$

$$\therefore 2\bar{d}=\bar{b}+\bar{c},\ 2\bar{e}=\bar{a}+\bar{c} \text{ and } 2\bar{f}=\bar{a}+\bar{b}$$

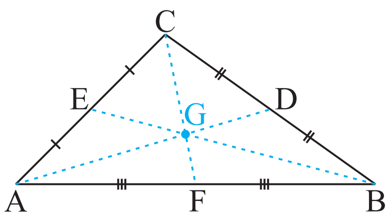


Fig 5.39

$\therefore 2\bar{d} + \bar{a} = \bar{a} + \bar{b} + \bar{c}$, *similarly* $2\bar{e} + \bar{b} = 2\bar{f} + \bar{c} = \bar{a} + \bar{b} + \bar{c}$

$$\therefore \frac{2\bar{d} + \bar{a}}{3} = \frac{2\bar{e} + \bar{b}}{3} = \frac{2\bar{f} + \bar{c}}{3} = \frac{\bar{a} + \bar{b} + \bar{c}}{3} = \bar{g} \text{ (say)}$$

Then we have $\bar{g} = \dfrac{\bar{a} + \bar{b} + \bar{c}}{3} = \dfrac{(2)\bar{d} + (1)\bar{a}}{2+1} = \dfrac{(2)\bar{e} + (1)\bar{b}}{2+1} = \dfrac{(2)\bar{f} + (1)\bar{c}}{2+1}$

If G is the point whose position vector is $\bar{g}$, then from the above equation it is clear that the point G lies on the medians AD, BE, CF and it divides each of the medians AD, BE, CF internally in the ratio 2 : 1.

Therefore, three medians are concurrent.

Ex. 6. Prove that the angle bisectors of a triangle are concurrent.

Solution :

Let A, B and C be vertices of a triangle. Let AD, BE and CF be the angle bisectors of the triangle ABC. Let $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$ and $\bar{f}$ be the position vectors of the points A, B, C, D, E and F respectively. Also AB = z BC = x AC = y. Now, the angle bisector AD meets the side BC at the point D. Therefore, the point D divides the line segment BC internally in the ratio AB : AC, that is $z : y$.

Hence, by section formula for internal division, we have $\bar{d} = \dfrac{z\bar{c} + y\bar{b}}{z+y}$

Similarly, we get

$$\bar{e} = \frac{x\bar{a} + z\bar{c}}{x+z} \quad \text{and} \quad \bar{f} = \frac{y\bar{b} + x\bar{a}}{y+x}$$

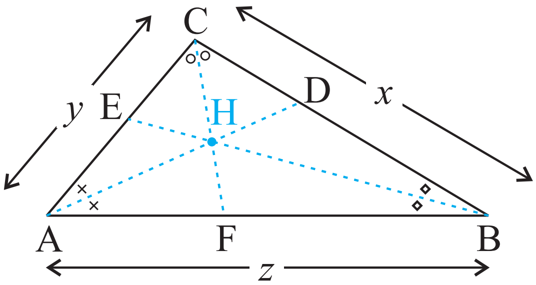


Fig 5.40

As $\bar{d} = \dfrac{z\bar{c} + y\bar{b}}{z+y}$

$\therefore$ $(z+y)\bar{d} = z\bar{c} + y\bar{b}$

i.e. $(z+y)\bar{d} + x\bar{a} = x\bar{a} + y\bar{b} + z\bar{c}$

similarly $(x+z)\bar{e} + y\bar{b} = x\bar{a} + y\bar{b} + z\bar{c}$

and $(x+y)\bar{f} + z\bar{c} = x\bar{a} + y\bar{b} + z\bar{c}$

$$\therefore \frac{(z+y)\bar{d} + x\bar{a}}{x+y+z} = \frac{(x+z)\bar{e} + y\bar{b}}{x+y+z} = \frac{(x+y)\bar{f} + z\bar{c}}{x+y+z} = \frac{x\bar{a} + y\bar{b} + z\bar{c}}{x+y+z} = \bar{h} \text{ (say)}$$

Then we have

$$\bar{h} = \frac{(y+z)\bar{d} + x\bar{a}}{(y+z)+x} = \frac{(x+z)\bar{e} + y\bar{b}}{(x+z)+y} = \frac{(x+y)\bar{f} + z\bar{c}}{(x+y)+z}$$

That is point H($\bar{h}$) divides AD in the ratio $(y + z) : x$, BE in the ratio $(x + z) : y$ and CF in the ratio $(x + y) : z$.

This shows that the point H is the point of concurrence of the angle bisectors AD, BE and CF of the triangle ABC, Thus, the angle bisectors of a triangle are concurrent and H is called incentre of the triangle ABC.

Ex. 7. Using vector method, find the incenter of the triangle whose vertices are A(0, 3, 0), B(0, 0, 4) and C(0, 3, 4).

Solution :

Let $\bar{a} = 3\hat{j}, \bar{b} = 4\hat{k}$ and $\bar{c} = 3\hat{j} + 4\hat{k}$

$\therefore \quad \overline{AB} = \bar{b} - \bar{a} = -3\hat{j} + 4\hat{k}, \ \overline{AC} = \bar{c} - \bar{a} = 4\hat{k}, \ \overline{BC} = \bar{c} - \bar{b} = 3\hat{j}$

$\therefore \quad |\overline{AB}| = 5, |\overline{AC}| = 4, |\overline{BC}| = 3$

If H ($\bar{h}$) is the incenter of triangle ABC then,

$$\therefore \quad \bar{h} = \frac{|\overline{BC}|\bar{a} + |\overline{AC}|\bar{b} + |\overline{AB}|\bar{c}}{|\overline{BC}| + |\overline{AC}| + |\overline{AB}|}$$

$$\therefore \quad \bar{h} = \frac{3(3j) + 4(4k) + 5(3j + 4k)}{3 + 4 + 5}$$

$$\therefore \quad = \frac{9j + 16k + 15j + 20k}{12}$$

$$\therefore \quad \bar{h} = \frac{24\hat{j} + 36\hat{k}}{12}$$

$$\therefore \quad \bar{h} = 2\hat{j} + 3\hat{k}$$

And $\quad H \equiv (0, 2, 3)$

Note : In ΔABC,

1) P.V. of Centroid is given by $\dfrac{\bar{a} + \bar{b} + \bar{c}}{3}$

2) P.V. of Incentre is given by $\dfrac{|\overline{AB}|\bar{c} + |\overline{BC}|\bar{a} + |\overline{AC}|\bar{b}}{|\overline{AB}| + |\overline{BC}| + |\overline{AC}|}$

3) P.V. of Orthocentre is given by $\dfrac{\tan A\bar{a} + \tan B\bar{b} + \tan C\bar{c}}{\tan A + \tan B + \tan C}$ (Verify)

Ex. 8. If $4\hat{i} + 7\hat{j} + 8\hat{k}, 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $2\hat{i} + 5\hat{j} + 7\hat{k}$ are the position vectors of the vertices A, B and C respectively of triangle ABC. Find the position vector of the point in which the bisector of $\angle$A meets BC.

Solution : $\bar{a} = 4\hat{i} + 7\hat{j} + 8\hat{k}, \ \bar{b} = 2\hat{i} + 3\hat{j} + 4\hat{j}, \ \bar{c} = 2i + 5j + 7k$

$\overline{AC} = \bar{c} - \bar{a} = -2\hat{i} - 2\hat{j} - \hat{k}$

$\overline{AB} = \bar{b} - \bar{a} = -2\hat{i} - 4\hat{j} - 4\hat{k}$

$\therefore |\overline{AB}| = \sqrt{4 + 16 + 16} = 6$ units

$\therefore |\overline{AC}| = \sqrt{4 + 4 + 1} = 3$ units

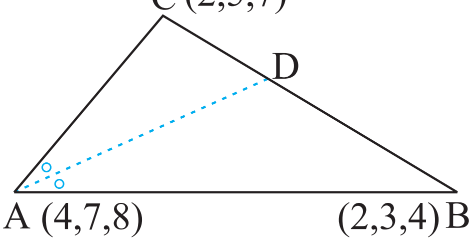


Fig 5.41

Let D be the point where angle bisector of $\angle$A meets BC.

D divides BC in the ratio AB : AC

i.e. $\bar{d} = \dfrac{|\overline{AB}|\bar{c} + |\overline{AC}|\bar{b}}{|\overline{AB}| + |\overline{AC}|}$

$= \dfrac{6\left(2\hat{i}+5\hat{j}+7\hat{k}\right)+3\left(2\hat{i}+3\hat{j}+4\hat{k}\right)}{6+3}$

$= \dfrac{\left(12\hat{i}+30\hat{j}+42\hat{k}\right)+\left(6\hat{i}+9\hat{j}+12\hat{k}\right)}{9}$

$= \dfrac{18\hat{i}+39\hat{j}+54\hat{k}}{9}$

$\therefore \bar{d} = 2\hat{i}+\dfrac{13}{3}\hat{j}+6\hat{k}$

Ex.9. If G(a, 2, -1) is the centroid of the triangle with vertices P(1, 3, 2), Q(3, b, -4) and R(5, 1, c) then find the values of a, b and c.

Solution : As $G(\bar{g})$ is centroid of ΔPQR $\bar{g} = \dfrac{\bar{p}+\bar{q}+\bar{r}}{3}$

$$a\hat{i}+2\hat{j}-\hat{k} = \frac{\left(\hat{i}+3\hat{j}+2\hat{k}\right)+\left(3\hat{i}+b\hat{j}-4\hat{k}\right)+\left(5\hat{i}+\hat{j}+c\hat{k}\right)}{3} = \frac{(1+3+5)\hat{i}+(3+b+1)\hat{j}+(2-4+c)\hat{k}}{3}$$

by equality of vectors $\quad a = \dfrac{1+3+5}{3} = \dfrac{9}{3} = 3 \quad\quad \therefore a = 3$

$\therefore 2 = \dfrac{3+b+1}{3} \quad \therefore 6 = 4+b \quad \therefore b = 2$

$\therefore -1 = \dfrac{2-4+c}{3} \quad \therefore -3 = -2+c \quad \therefore c = -1$.

Ex. 10. Find the centroid of tetrahedron with vertices A(3, -5, 7), B(5, 4, 2), C(7, -7, -3), D(1, 0, 2) ?

Solution : Let $\bar{a} = 3\hat{i}-5\hat{j}+7\hat{k}, \bar{b} = 5\hat{i}+4\hat{j}+2\hat{k}, \bar{c} = 7\hat{i}-7\hat{j}-3\hat{k}, \bar{d} = \hat{i}+2\hat{k}$ be position vectors of vertices A, B, C & D.

By centroid formula, centroid G($\bar{g}$) is given by

$\bar{g} = \dfrac{\bar{a}+\bar{b}+\bar{c}+\bar{d}}{4}$

$= \dfrac{\left(3\hat{i}-5\hat{j}+7\hat{k}\right)+\left(5\hat{i}+4\hat{j}+2\hat{k}\right)+\left(7\hat{i}-7\hat{j}-3\hat{k}\right)+\left(\hat{i}+2\hat{k}\right)}{4}$

$= \dfrac{(3+5+7+1)\hat{i}+(-5+4-7+0)\hat{j}+(7+2-3+2)\hat{k}}{4} = \dfrac{16\hat{i}-8\hat{j}+8\hat{k}}{4}$

$= 4\hat{i}-2\hat{j}+2\hat{k}$

Therefore, centroid of tetrahedron is G ≡ (4, –2, 2).

Ex. 11. Find the ratio in which point P divides AB and CD where A(2, –3, 4), B(0, 5, 2), C(–1, 5, 3) and D (2, –1, 3). Also, find its coordinates.

Solution : Let point P divides AB in ratio m : 1 and CD in ratio n : 1.

By section formula,

$$P \equiv \left(\frac{2}{m+1}, \frac{5m-3}{m+1}, \frac{2m+4}{m+1}\right) \equiv \left(\frac{2n-1}{n+1}, \frac{n+5}{n+1}, \frac{3n+3}{n+1}\right)$$

Equating z-coordinates

$$\therefore \quad \frac{2m+4}{m+1} = \frac{3n+3}{n+1}$$

$$\frac{2m+4}{m+1} = \frac{3(n+1)}{(n+1)}$$

$2m + 4 = 3(m + 1)$

$2m + 4 = 3m + 3$

$1 = m$

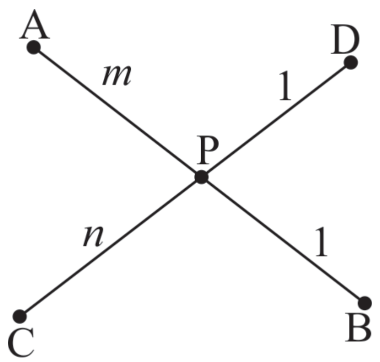


Fig 5.42

Also, by equating x-coordinates

$$\frac{2}{m+1} = \frac{2n-1}{n+1}$$

$$\frac{2}{1+1} = \frac{2n-1}{n+1} (m=1)$$

$n + 1 = 2n - 1$

$2 = n$

P divides AB in ratio m : 1 *i.e.* 1 : 1 and CD in the ratio n : 1 *i.e.* 2 : 1.

$$P \equiv \left(\frac{2}{1+1}, \frac{5-3}{1+1}, \frac{2+4}{1+1}\right) \equiv (1,1,3).$$

Ex. 12. In a triangle ABC, D and E are points on BC and AC respectively, such that BD = 2 DC and AE = 3 EC. Let P be the point of intersection of AD and BE. Find BP/PF using vector methods.

Solution : Let $\bar{a}$, $\bar{b}$, $\bar{c}$ be the position vectors of A, B and C respectively with respect to some origin.

D divides BC in the ratio 2 : 1 and E divides AC in the ratio 3 : 1.

$$\therefore \quad \bar{d} = \frac{\vec{b}+2\vec{c}}{3} \quad \bar{e} = \frac{\vec{a}+3\vec{c}}{4}.$$

Let point of intersection P of AD and BE divides BE

in the ratio k : 1 and AD in the ratio m : 1, then position

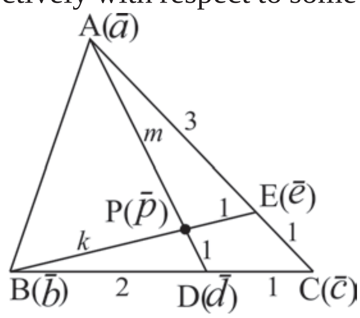


Fig 5.43

vectors of P in these two cases are $\dfrac{\vec{b}+k\left(\dfrac{\vec{a}+3\vec{c}}{4}\right)}{k+1}$ and $\dfrac{\vec{a}+m\left(\dfrac{\vec{b}+2\vec{c}}{3}\right)}{m+1}$ and respectively.

Equating the position vectors of P we get,

$$\frac{k}{4(k+1)}\bar{a}+\frac{1}{k+1}\bar{b}+\frac{3k}{4(k+1)}\bar{c} = \frac{1}{m+1}\bar{a}+\frac{m}{3(m+1)}\bar{b}+\frac{2m}{3(m+1)}\bar{c}$$

$$\therefore \frac{k}{4(k+1)}=\frac{1}{m+1} \quad \text{... (1)}$$

$$\frac{1}{k+1}=\frac{m}{3(m+1)} \quad \text{... (2)}$$

$$\frac{3k}{4(k+1)}=\frac{2m}{3(m+1)} \quad \text{... (3)}$$

Dividing (3) by (2) we get,

$$\frac{3k}{4}=2 \text{ i.e. } k=\frac{8}{3} \quad \text{therefore} \quad \frac{BP}{PF}=k:1=8:3$$

Exercise 5.2

1. Find the position vector of point R which divides the line joining the points P and Q whose position vectors are $2\hat{i}-\hat{j}+3\hat{k}$ and $-5\hat{i}+2\hat{j}-5\hat{k}$ in the ratio 3 : 2 (i) internally (ii) externally.
2. Find the position vector of mid-point M joining the points L (7, -6,12) and N (5, 4, -2).
3. If the points A(3, 0, p), B (-1, q, 3) and C(-3, 3, 0) are collinear, then find
   (i) The ratio in which the point C divides the line segment AB.
   (ii) The values of p and q.
4. The position vector of points A and B are $6\bar{a}+2\bar{b}$ and $\bar{a}-3\bar{b}$. If the point C divides AB in the ratio 3 : 2 then show that the position vector of C is $3\bar{a}-\bar{b}$.
5. Prove that the line segments joining mid-point of adjacent sides of a quadrilateral form a parallelogram.
6. D and E divide sides BC and CA of a triangle ABC in the ratio 2 : 3 respectively. Find the position vector of the point of intersection of AD and BE and the ratio in which this point divides AD and BE.
7. Prove that a quadrilateral is a parallelogram if and only if its diagonals bisect each other.
8. Prove that the median of a trapezium is parallel to the parallel sides of the trapezium and its length is half the sum of parallel sides.
9. If two of the vertices of the triangle are A(3, 1, 4) and B(-4, 5, -3) and the centroid of a triangle is G(-1, 2, 1), then find the coordinates of the third vertex C of the triangle.
10. In ΔOAB, E is the mid-point of OB and D is the point on AB such that AD : DB = 2 : 1. If OD and AE intersect at P, then determine the ratio OP : PD using vector methods.
11. If the centroid of a tetrahedron OABC is (1, 2, –1) where A = (a, 2, 3), B = (1, b, 2), C = (2, 1, c) respectively, find the distance of P (a, b, c) from the origin.
12. Find the centroid of tetrahedron with vertices K(5, –7, 0), L(1, 5, 3), M(4, – 6, 3), N(6, –4, 2) ?

5.3 Product of vectors :

The product of two vectors is defined in two different ways. One form of product results in a scalar quantity while other form gives a vector quantity. Let us study these products and interpret them geometrically.

Angle between two vectors :

When two non zero vectors $\bar{a}$ and $\bar{b}$ are placed such that their initial points coincide, they form an angle θ of measure $0 \le \theta \le \pi$.

Angle between $\bar{a}$ and $\bar{b}$ is also denoted as $\widehat{\bar{a}\ \bar{b}}$. The angle between the collinear vectors is 0 if they point in the same direction and π if they are in opposite directions.

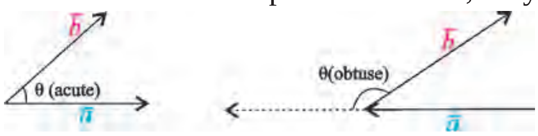


Fig 5.44

5.3.1 Scalar product of two vectors :

The scalar product of two non-zero vectors $\bar{a}$ and $\bar{b}$ is denoted by $\bar{a}\cdot\bar{b}$, and is defined as $\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|$ cos θ, where θ is the angle between $\bar{a}$ and $\bar{b}$.

$\bar{a}\cdot\bar{b}$ is a real number, that is, a scalar. For this reason, the dot product is also called a scalar product.

Note :

1) If either $\bar{a} = 0$ or $\bar{b} = 0$ then θ is not defined and in this case, we define $\vec{a}\cdot\vec{b} = 0$.

2) If θ = 0, then $\bar{a}\cdot\bar{b} = |\bar{a}|\,|\bar{b}|\ \cos 0 = |\bar{a}|\,|\bar{b}|$. In particular, $\bar{a}\cdot\bar{a} = |\bar{a}|^2$, as θ = 0 .

3) If θ = π, then $\bar{a}\cdot\bar{b} = |\bar{a}|\,|\bar{b}|\ \cos\pi = -|\bar{a}|\,|\bar{b}|$.

4) If $\bar{a}$ and $\bar{b}$ are perpendicular or orthogonal then θ = π/2

Conversely if $\bar{a}\cdot\bar{b} = 0$ then either $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$ or θ = π/2.

Also, $\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\theta = |\bar{b}||\bar{a}|\cos\theta = \bar{b}\cdot\bar{a}$.

5) Dot product is distributive over vector addition. If $\bar{a}$, $\bar{b}$, $\bar{c}$ are any three vectors, then

$\bar{a}\cdot(\bar{b}+\bar{c}) = \bar{a}\cdot\bar{b} + \bar{a}\cdot\bar{c}$.

6) If $\bar{a}$ and $\bar{b}$ are vectors and *m*, *n* are scalars, then

(i) $(m\bar{a})\cdot(n\bar{b}) = mn(\bar{a}\cdot\bar{b})$

(ii) $(m\bar{a})\cdot\bar{b} = m(\bar{a}\cdot\bar{b}) = \bar{a}\cdot(m\bar{b})$

7) $|\bar{a}\cdot\bar{b}| \le |\bar{a}||\bar{b}|$ (This is known as Cauchy Schwartz Inequality).

5.3.2 Finding angle between two vectors :

Angle θ, $(0 \le \theta \le \pi)$ between two non-zero vectors $\bar{a}$ and $\bar{b}$ is given by $\cos\theta = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$, that is

$$\theta = \cos^{-1}\left(\frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}\right).$$

Note :

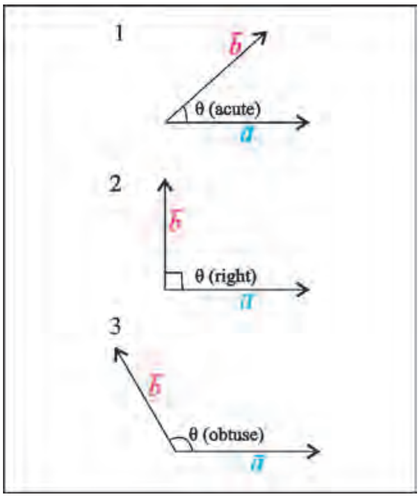


Fig 5.45

1) If $0 \le \theta < \dfrac{\pi}{2}$, then $\cos\theta = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} > 0$, that is $\bar{a}\cdot\bar{b} > 0$.

2) If $\theta = \dfrac{\pi}{2}$, then $\cos\theta = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} = 0$, that is $\bar{a}\cdot\bar{b} = 0$.

3) If $\dfrac{\pi}{2} < \theta \le \pi$, then $\cos\theta = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} < 0$, that is $\bar{a}\cdot\bar{b} < 0$.

4) In particular scalar product of $\hat{i}$, $\hat{j}$, $\hat{k}$ vectors are

(i) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ and (ii) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$.

The scalar product of vectors $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\bar{a}\cdot\bar{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)\cdot\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) = a_1b_1 + a_2b_2 + a_3b_3$$

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2b_3^2}}$$

5.3.3 Projections :

$\overline{PQ}$ and $\overline{PR}$ represent the vectors $\bar{a}$ and $\bar{b}$ with same initial point P. If M is the foot of perpendicular from R to the line containing $\overline{PQ}$ then $|\overline{PS}|$ is called the scalar projection of $\bar{b}$ on $\bar{a}$. We can think of it as a shadow of $\bar{b}$ on $\bar{a}$, when sun is overhead.

Scalar Projection of $\bar{b}$ on $\bar{a}$ $= |\overline{PS}| = |\bar{b}|\cos\theta = \dfrac{|\bar{a}||\bar{b}|\cos\theta}{|\bar{a}|}$

Scalar Projection of $\bar{b}$ on $\bar{a} = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}|}$

Vector Projection of $\bar{b}$ on $\bar{a}$ $= \overline{PS}$

$= |\overline{PS}|\hat{a}$, where $\hat{a}$ is unit vector along $\bar{a}$

$= \left(\dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}|}\right)\hat{a}$

$= \left(\dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}|}\right)\dfrac{\bar{a}}{|\bar{a}|}$

Vector Projection of $\bar{b}$ on $\bar{a} = \left(\bar{a}\cdot\bar{b}\right)\dfrac{\bar{a}}{|\bar{a}|^2}$

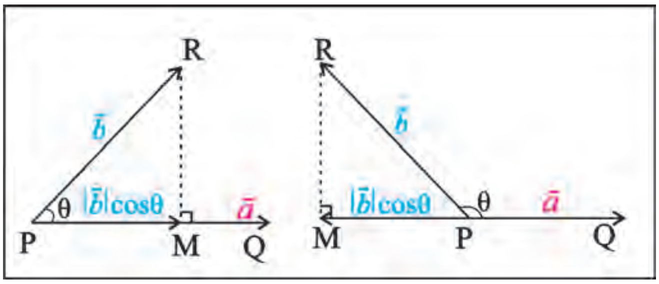


Fig 5.46

5.3.4 Direction Angles and Direction Cosines :

The direction angles of a non-zero vector $\bar{a}$ are the angles α, β,and γ $\left(\in[0,\pi]\right)$ that $\bar{a}$ makes with the positive X-, Y- and Z-axes respectively. These angles completely determine the direction of the vector $\bar{a}$.

The cosines of these direction angles, that is cos α, cos β, cos γ are called the direction cosines (abbreviated as d.c.s) of vector $\bar{a}$

As α is angle between $\hat{i}$ (unit vector) along X-axis and $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ then.

$$\cos\alpha=\frac{\bar{a}\cdot\hat{i}}{|\bar{a}||\hat{i}|}=\frac{\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)\cdot\hat{i}}{|\bar{a}|(1)}=\frac{a_1}{|\bar{a}|},\ \text{Similarly}\ \cos\beta=\frac{a_2}{|\bar{a}|}\ \text{and}\ \cos\gamma=\frac{a_3}{|\bar{a}|},$$

where $|\bar{a}|=\sqrt{a_1^{\,2}+a_2^{\,2}+a_3^{\,2}}$.

By squaring and adding, we get $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\dfrac{a_1^{\,2}}{|\bar{a}|^2}+\dfrac{a_2^{\,2}}{|\bar{a}|^2}+\dfrac{a_3^{\,2}}{|\bar{a}|^2}$.

As $|\bar{a}|^2=a_1^{\,2}+a_2^{\,2}+a_3^{\,2}$

$$\therefore\ \boxed{\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1}$$

Also, $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$$=|\bar{a}|\cos\alpha\hat{i}+|\bar{a}|\cos\beta\hat{j}+|\bar{a}|\cos\gamma\hat{k}$$

$$=|\bar{a}|\left(\cos\alpha\hat{i}+\cos\beta\hat{j}+\cos\gamma\hat{k}\right)$$

That is, $\dfrac{\bar{a}}{|\bar{a}|}=\cos\alpha\hat{i}+\cos\beta\hat{j}+\cos\gamma\hat{k}=\hat{a}$

Which means that the direction cosines of $\bar{a}$, are components of the unit vector in the direction of $\bar{a}$

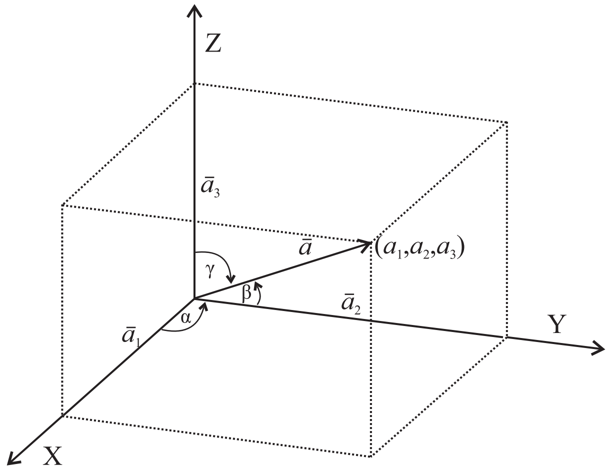


Fig 5.47

Direction cosines (d.c.s) of any line along a vector $\bar{a}$ has same direction cosines as that of $\bar{a}$.

Direction cosines are generally denoted by l, m, n , where l = cos α, m = cos β, n = cos γ.

As the unit vectors along X, Y- and Z- axes are $\hat{i}$, $\hat{j}$, $\hat{k}$. Then $\hat{i}$ makes the direction angles $0,\dfrac{\pi}{2},\dfrac{\pi}{2}$ so its direction cosines are $\cos 0,\cos\dfrac{\pi}{2},\cos\dfrac{\pi}{2}$ that is $1, 0, 0$. Similarly direction cosines of Y-and Z- axes are $0,1,0$ and $0,0,1$ respectively.

Let $\overline{OL}$ and $\overline{OL'}$ be the vectors in the direction of line LL´. If α, β,and γ are direction angles of $\overline{OL}$ then the direction angles of $\overline{OL'}$ are π – α, π – β,and π – γ. Therefore, direction cosines of $\overline{OL}$ are cos α, cos β, cos γ *i.e. l, m, n* whereas direction cosines of $\overline{OL'}$ are cos(π – α), cos(π – β) and cos (π – γ). *i.e.* -cos α, -cos β and -cos γ. *i.e.* -*l*,-*m*,-*n*. Therefore direction cosines of line LL' are same as that of vectors $\overline{OL}$ or $\overline{OL'}$ in the direction of line LL'. i.e. either *l,m,n* or *-l,-m,-n*. As $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ so $l^2 + m^2 + n^2 = 1$.

Direction ratios :

Any 3 numbers which are proportional to direction cosines of the line are called the direction ratios (abbreviated as *d.r.s*) of the line. Generally the direction ratios are denoted by *a, b, c*.

If *l, m, n* are the direction cosines and *a, b, c* are direction ratios then

$a = \lambda l,\ b = \lambda m,\ c = \lambda n$, for some $\lambda \in R$.

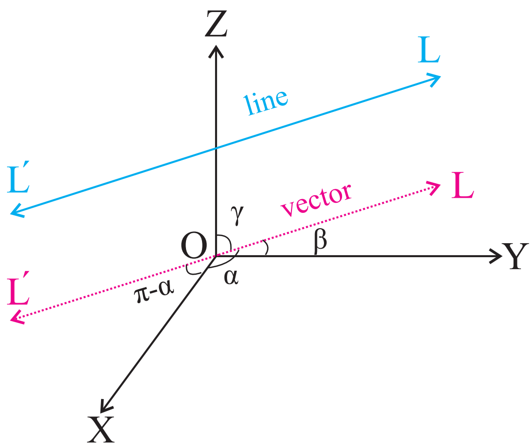


Fig 5.48

For Example.: If direction cosines of the line are $0,\frac{1}{2},\frac{\sqrt{3}}{2}$ then $0,1,\sqrt{3}$ or $0,\sqrt{3},3$ or $0,2\sqrt{3},6$ are also direction ratios of the same line.

Note : A line has infinitely many direction ratios but unique direction cosines.

Relation between direction ratios and direction cosines :

Let a,b,c be direction ratios and l,m,n be direction cosines of a line.

By definition of d.r.s, $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\lambda$ i.e. $l = \lambda a,\ m = \lambda b,\ n = \lambda c$.

But $l^2 + m^2 + n^2 = 1$

$\therefore \quad (\lambda a)^2 + (\lambda b)^2 + (\lambda c)^2 = 1$

$\therefore \quad \lambda^2(a^2 + b^2 + c^2 = 1$

$\therefore \quad \lambda = \pm\frac{1}{\sqrt{a^2+b^2+c^2}}$

$\therefore \quad l=\pm\frac{a}{\sqrt{a^2+b^2+c^2}},\ m=\pm\frac{b}{\sqrt{a^2+b^2+c^2}}$ and $n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}}$.

Note that direction cosines are similar to the definition of unit vector, that is if $\bar{x}=a\hat{i}+b\hat{j}+c\hat{k}$ be any vector (d.r.s) then $\hat{x}=\pm\frac{\bar{x}}{|\bar{x}|}=\pm\frac{a\hat{i}+b\hat{j}+c\hat{k}}{\sqrt{a^2+b^2+c^2}}$ is unit vector (d.c.s) along $\bar{x}$.

Solved Examples

Ex.1. Find $\bar{a}\cdot\bar{b}$ if $|\bar{a}|=3, |\bar{b}|=\sqrt{6}$, the angle between $\bar{a}$ and $\bar{b}$ is 45°.

Solution : $\bar{a}\cdot\bar{b}=|\bar{a}|\,|\bar{b}|\cos\theta=(3)(\sqrt{6})\cos 45^\circ = 3\sqrt{6}\left(\frac{\sqrt{2}}{2}\right)=\frac{3}{2}\cdot 2\sqrt{3}=3\sqrt{3}$

Ex.2. If $\bar{a}=3\hat{i}+4\hat{j}-5\hat{k}$ and $\bar{b}=3\hat{i}-4\hat{j}-5\hat{k}$

(i) find $\bar{a}\cdot\bar{b}$

(ii) the angle between $\bar{a}$ and $\bar{b}$.

(iii) the scalar projection of $\bar{a}$ in the direction of $\bar{b}$.

(iv) the vector projection of $\bar{b}$ along $\bar{a}$.

Solution : Here $\bar{a}=3\hat{i}+4\hat{j}-5\hat{k}$ and $\bar{b}=3\hat{i}-4\hat{j}-5\hat{k}$

i) $\bar{a}\cdot\bar{b}=(3)(3)+(4)(-4)+(-5)(-5)=9-16+25=18$

ii) $|\bar{a}|=\sqrt{9+16+25}=\sqrt{50},\quad |\bar{b}|=\sqrt{9+16+25}=\sqrt{50}$

The angle between $\bar{a}$ and $\bar{b}$ is $\cos\theta=\frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|}=\frac{18}{50}\ \therefore \theta=\cos^{-1}\left(\frac{18}{50}\right)$

iii) The scalar projectio of $\bar{a}$ in the direction of $\bar{b}$ is $\frac{\bar{a}\cdot\bar{b}}{|\bar{b}|}=\frac{18}{\sqrt{50}}=\frac{18}{5\sqrt{2}}$.

iv) The vector projection of $\bar{b}$ along is $\bar{a}$ is $=(\bar{a}\cdot\bar{b})\frac{\bar{a}}{|\bar{a}|^2}=\frac{18}{50}(3\hat{i}+4\hat{j}-5\hat{k})=\frac{9}{25}(3\hat{i}+4\hat{j}-5\hat{k})$.

Ex.3. Find the value of a for which the vectors $3\hat{i}+2\hat{j}+9\hat{k}$ and $\hat{i}+a\hat{j}+3\hat{k}$ are (i) perpendicular (ii) parallel

Solution : Let $\bar{p}=3\hat{i}+2\hat{j}+9\hat{k}$ and $\bar{q}=\hat{i}+a\hat{j}+3\hat{k}$

(i) The two vectors are perpendicular if $\bar{p}\cdot\bar{q}=0$ *i.e.* $(3\hat{i}+2\hat{j}+9\hat{k})\cdot(\hat{i}+a\hat{j}+3\hat{k})=0$

i.e. 3(1) + 2(a) + 9(3) = 0. i.e. 2a + 30 = 0 or a = -15.

(ii) The two vectors are parallel if $\frac{3}{1}=\frac{2}{a}=\frac{9}{3}$ *i.e* $3a=2$ *i.e.* $a=\frac{2}{3}$.

Ex.4. If $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}+2\hat{k}$ find the angle between the vectors $2\vec{a}+\vec{b}$ and $\vec{a}+2\vec{b}$.

Solution : $2\bar{a}+\bar{b}=2(\hat{i}+2\hat{j}-3\hat{k})+(3\hat{i}-\hat{j}+2\hat{k})=5\hat{i}+3\hat{j}-4\hat{k}='\bar{m}'\ (say)$

$\bar{a}+2\bar{b}=(\hat{i}+2\hat{j}-3\hat{k})+2(3\hat{i}-\hat{j}+2\hat{k})=7\hat{i}+\hat{k}='\bar{n}'(say)$

$\therefore \bar{m}\cdot\bar{n}=(2\bar{a}+\bar{b})\cdot(\bar{a}+2\bar{b})=(5\hat{j}+3\hat{j}-4\hat{k})\cdot(7\hat{i}+\hat{k})$

= (5) (7) + (3) (0) + (-4) (1) = 31

$$|\overline{m}| = \sqrt{(5)^2+(3)^2+(-4)^2} = \sqrt{50}$$

$$|\overline{n}| = \sqrt{(7)^2+(0)^2+(1)^2} = \sqrt{50}$$

If θ is the angle between $\overline{m}$ and $\overline{n}$ then

$$\cos\theta = \frac{\overline{m}\cdot\overline{n}}{|\overline{m}|\,|\overline{n}|} = \frac{31}{\sqrt{50}\times\sqrt{50}} = \frac{31}{50}$$

$$\therefore \theta = \cos^{-1}\left(\frac{31}{50}\right)$$

Ex 5. : If a line makes angle 90°, 60° and 30° with the positive direction of X, Yand Z axes respectively, find its direction cosines.

Solution : Let the d.c.s. of the lines be l, m, n then $l = \cos 90° = 0$, $m = \cos 60° = \frac{1}{2}$
$n = \cos 30° = \frac{\sqrt{3}}{2}$. Therefore, l, m, n are $0, \frac{1}{2}, \frac{\sqrt{3}}{2}$

Ex. 6 : Find the vector projection of $\vec{PQ}$ on $\vec{AB}$ where P, Q, A, B are the points (-2, 1, 3), (3, 2, 5) (4, -3, 5) and (7, -5, -1) respectively.

Solution : Let the position vectors of P, Q, A, B are $\overline{p}, \overline{q}, \overline{a}, \overline{b}$ respectively

$$\overline{p} = -2\hat{i}+\hat{j}+3\hat{k},\ \overline{q} = 3\hat{i}+2\hat{j}+5\hat{k}$$
$$\overline{a} = 4\hat{i}-3\hat{j}+5\hat{k},\ \overline{b} = 7\hat{i}-5\hat{j}-\hat{k}$$

$$\therefore \quad \vec{PQ} = \overline{q}-\overline{p} = \left(3\hat{i}+2\hat{j}+5\hat{k}\right)-\left(-2\hat{i}+\hat{j}+3\hat{k}\right) = 5\hat{i}+\hat{j}+2\hat{k}$$
$$\text{and } \vec{AB} = \overline{b}-\overline{a} = \left(7\hat{i}-5\hat{j}-\hat{k}\right)-\left(4\hat{i}-3\hat{j}+5\hat{k}\right) = 3\hat{i}-2\hat{j}-6\hat{k}$$

$\therefore$ Vector Projection of $\overline{PQ}$ on $\overline{AB}$

$$= \frac{\overline{PQ}\cdot\overline{AB}}{|\overline{AB}|^2}\overline{AB} = \frac{(5)(3)+(1)(-2)+(2)(-6)}{(3)^2+(-2)^2+(-6)^2}\overline{AB}$$

$$= \frac{1}{49}\left(3\hat{i}-2\hat{j}-6\hat{k}\right) = \frac{3}{49}\hat{i}-\frac{2}{49}\hat{j}-\frac{6}{49}\hat{k}$$

Ex. 7 : Find the values of λ for which the angle between the vectors

$\vec{a} = 2\lambda^2\hat{i}+4\lambda\hat{j}+\hat{k}$ and $\vec{b} = 7\hat{i}-2\hat{j}+\lambda\hat{k}$ is obtuse.

Solution : If θ is the angle between $\vec{a}$ and $\vec{b}$, then $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$

If θ is obtuse then $\cos\theta < 0$

$$\therefore \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} < 0 \text{ i.e. } \vec{a}\cdot\vec{b} < 0 \qquad \left[\because |\vec{a}||\vec{b}| > 0\right]$$

$$\therefore \left(2\lambda^2\hat{i} + 4\lambda\hat{j} + \hat{k}\right)\cdot\left(7\hat{i} - 2\hat{j} + \lambda\hat{k}\right) < 0$$

$$\therefore\ 14\lambda^2 - 8\lambda + \lambda < 0 \text{ i.e. } 14\lambda^2 - 7\lambda < 0$$

$$\therefore\ 7\lambda(2\lambda - 1) < 0 \text{ i.e. } \lambda\left(\lambda - \frac{1}{2}\right) < 0 \text{ i.e. } 0 < \lambda < \frac{1}{2}$$

Thus the angle between $\bar{a}$ and $\bar{b}$ is obtuse if $0 < \lambda < \frac{1}{2}$

Ex. 8 Find the direction cosines of the vector $2\hat{i} + 2\hat{j} - \hat{k}$

Solutions : Let $|\bar{a}| = 2\hat{i} + 2\hat{j} - \hat{k}$

$$\therefore |\bar{a}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{9} = 3$$

$$\therefore \hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{2\hat{i} + 2\hat{j} + \hat{k}}{3} = \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$$

$\therefore$ The direction cosines of $\bar{a}$ are $\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}$

Ex. 9 : Find the position vector of a point P such that AB is inclined to X axis at 45^0 and to Y axis at 60^0 and OP = 12 units.

Solution : We have $l = \cos 45^\circ = \frac{1}{\sqrt{2}}$, $m = \cos 60^\circ = \frac{1}{2}$, $n = \cos\gamma$

Now $l^2 + m^2 + n^2 = 1$

$$\therefore \frac{1}{2} + \frac{1}{4} + \cos^2\gamma = 1, \text{ i.e. } \cos^2\gamma = \frac{1}{4}, \text{ i.e. } n = \cos\gamma = \pm\frac{1}{2}$$

$$\text{Now } \vec{r} = |\vec{r}|\hat{r} = |\vec{r}|\left(l\hat{i} + m\hat{j} + n\hat{k}\right) = 12\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} \pm \frac{1}{2}\hat{k}\right)$$

Hence $\vec{r} = 6\sqrt{2}\hat{i} + 6\hat{j} \pm 6\hat{k}$

Ex. 10 A line makes angles of measure 45º and 60º with the positive direction of the Y and Z axes respectively. Find the angle made by the line with the positive directions of the X-axis.

Let α , β , γ be the angles made by the line with positive direction of X, Y and Z axes respectively. Given $\beta = 45^\circ$ and $\gamma = 60^\circ$.

Now $\cos\beta = \cos 45^\circ$ and $\cos\gamma = \cos 60^\circ = \frac{1}{2}$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$\therefore \quad \cos^2\alpha + \frac{1}{2} + \frac{1}{4} = 1$

$\therefore \quad \cos^2\alpha = \frac{1}{4}$

$\therefore \quad \cos\alpha = \pm\frac{1}{2}$

$\therefore \quad \alpha = 60^\circ$ or 120° There are two lines satisfying given conditions. Their direction angles are 45º, 60º, 60º and 45º, 60º , 120º

Ex. 11. A line passes through the points (6, -7, –1) and (2, -3, 1). Find the direction ratios and the direction cosines of the line so that the angle α is acute.

Solution : Let A(6, -7, -1) and B(2, -3, 1) be the given points. So $\bar{a} = 6\hat{i} - 7\hat{j} - \hat{k}, \bar{b} = 2\hat{i} - 3\hat{j} + \hat{k}$

$\overrightarrow{AB} = \bar{b} - \bar{a} = (2-6)\hat{i} + (-3+7)\hat{j} + (1+1)\hat{k} = -4\hat{i} + 4\hat{j} + 2\hat{k}$

the direction ratios of $\overrightarrow{AB}$ are –4, 4, 2 .

Let the direction cosines of $\overrightarrow{AB}$ be -4k, 4k, 2k. Then

$$(-4k)^2 + (4k)^2 + (2k)^2 = 1$$

i.e. $16k^2 + 16k^2 + 4k^2 = 1$ *i.e.* $36k^2 = 1$ *i.e.* $k = \pm\frac{1}{6}$

Since the line AB is so directed that the angle α which it makes with the x-Axis is acute,

$\therefore \quad \cos\alpha = -4k > 0$

$\therefore \quad$ As $k < 0 \quad \therefore k = -\frac{1}{6}$

$\therefore \quad$ the direction cosines of $\overrightarrow{AB}$ are $-4\left(-\frac{1}{6}\right), 4\left(-\frac{1}{6}\right), 2\left(-\frac{1}{6}\right)$ *i.e.* $\frac{2}{3}, \frac{-2}{3}, \frac{-1}{3}$

Ex. 12 Prove that the altitudes of a triangle are concurrent.

Solution : Let A, B and C be the vertices of a triangle

Let AD, BE and CF be the altitudes of the triangle ABC, therefore AD $\perp$ BC, BE $\perp$ AC, CF $\perp$ AB.

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}$ be the position vectors of *A, B, C, D, E, F* respectively. Let P be the point of intersection of the altitudes *AD* and *BE* with $\bar{p}$ as the position vector.

Therefore, $\overline{AP} \perp \overline{BC}, \overline{BP} \perp \overline{AC}$ ....(1)

To show that the altitudes *AD, BE* and *CF* are concurrent, it is sufficient to show that the altitude CF passes through the point *P*. We will have to prove that $\overline{CF}$ and $\overline{CP}$ are collinear vectors. This can be achieved by showing $\overline{CP} \perp \overline{AB}$

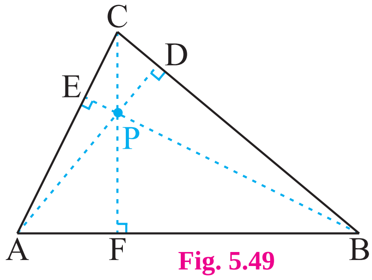


Fig. 5.49

Now from (1) we have

$$\overline{AP} \perp \overline{BC} \quad \text{and } \overline{BP} \perp \overline{AC}$$

$$\overline{AP} \perp \overline{BC} = 0 \quad \text{and } \overline{BP} \perp \overline{AC} = 0$$

$$\therefore (\bar{p} - \bar{a}).(\bar{c} - \bar{b}) = 0 \text{ and } (\bar{p} - \bar{b}).(\bar{c} - \bar{a}) = 0$$

$$\therefore \bar{p}.\bar{c} - \bar{p}.\bar{b} - \bar{a}.\bar{c} + \bar{a}.\bar{b} = 0 \qquad ..... (2)$$

$$\bar{p}.\bar{c} - \bar{p}.\bar{a} - \bar{b}.\bar{c} + \bar{b}.\bar{a} = 0 \qquad ...... (3)$$

Therefore, subtracting equation (2) from equation (3), we get

$$-\bar{p}.\bar{a} + \bar{p}.\bar{b} - \bar{b}.\bar{c} + \bar{a}.\bar{c} = 0 \qquad (\text{Since } \bar{a}.\bar{b} = \bar{b}.\bar{a})$$

$\therefore \quad \bar{p}(\bar{b} - \bar{a}) - \bar{c}(\bar{b} - \bar{a}) = 0$

$\therefore \quad (\bar{p} - \bar{c}) \cdot (\bar{b} - \bar{a}) = 0$

$\therefore \quad \overline{CP} \cdot \overline{AB} = 0$

$\therefore \quad \overline{CP} \perp \overline{AB}$

Hence the proof.

Exercise 5.3

1. Find two unit vectors each of which is perpendicular to both $\bar{u}$ and $\bar{v}$, where $\bar{u} = 2\hat{i} + \hat{j} - 2\hat{k}, \bar{v} = \hat{i} + 2\hat{j} - 2\hat{k}$

2. If $\bar{a}$ and $\bar{b}$ are two vectors perpendicular to each other, prove that $(\bar{a} + \bar{b})^2 = (\bar{a} - \bar{b})^2$

3. Find the values of *c* so that for all real *x* the vectors $xc\hat{i} - 6\hat{j} + 3\hat{k}$ *and* $x\hat{i} + 2\hat{j} + 2cx\hat{k}$ make an obtuse angle.

4. Show that the sum of the length of projections of $p\hat{i} + q\hat{j} + r\hat{k}$ on the coordinate axes, where $p = 2$, $q = 3$ and $r = 4$, is 9.

5. Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

6. Determine whether $\bar{a}$ and $\bar{b}$ are orthogonal, parallel or neither.

   $i)\ \bar{a} = -9\hat{i} + 6\hat{j} + 15\hat{k}, \bar{b} = 6\hat{i} - 4\hat{j} - 10\hat{k}$ $\qquad ii)\ \bar{a} = 2\hat{i} + 3\hat{j} - \hat{k}, \bar{b} = 5\hat{i} - 2\hat{j} + 4\hat{k}$

   $iii)\ \bar{a} = -\frac{3}{5}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{3}\hat{k}, \bar{b} = 5\hat{i} + 4\hat{j} + 3\hat{k}$ $\qquad iv)\ \bar{a} = 4\hat{i} - \hat{j} + 6\hat{k}, \bar{b} = 5\hat{i} - 2\hat{j} + 4\hat{k}$

7. Find the angle P of the triangle whose vertices are P(0, -1, -2), Q(3, 1, 4) and R(5, 7, 1).

8. If $\hat{p}, \hat{q}$ and $\hat{r}$ are unit vectors, find $i)\ \hat{p} \cdot \hat{q}$ $\quad ii)\ \hat{p} \cdot \hat{r}$ (see fig.5.50)

9. Prove by vector method that the angle subtended on semicircle is a right angle.

10. If a vector has direction angles 45°and 60° find the third direction angle.

11. If a line makes angles 90°, 135°, 45° with the X, Y and Z axes respectively, then find its direction cosines.

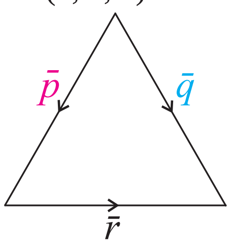


Fig.5.50

12. If a line has the direction ratios, 4, -12, 18 then find its direction cosines.

13. The direction ratios of $\overline{AB}$ are -2, 2, 1. If A = (4, 1, 5) and l (AB) = 6 units, find B.

14. Find the angle between the lines whose direction cosines l, m, n satisfy the equations

$5l+m+3n = 0$ and $5mn - 2nl + 6lm = 0$.

5.4.1 Vector Product of two vectors

In a plane, to describe how a line is tilting we used the notions of slope and angle of inclination. In space, we need to know how plane is tilting. We get this by multiplying two vectors in the plane together to get the third vector perpendicular to the plane. Third vector tell us inclination of the plane. The product we use for finding the third vector is called vector product.

Let $\bar{a}$ and $\bar{b}$ be two nonzero vectors in space. If $\bar{a}$ and $\bar{b}$ are not collinear, they determine a plane. We choose a unit vector $\hat{n}$ perpendicular to the plane by the right-hand rule. Which means $\hat{n}$ points in the way, right thumb points when our fingers curl through the angle from (See fig 5.51). Then we define a new vector $\bar{a}\times\bar{b}$ as $\bar{a}\times\bar{b} = |\bar{a}|\,|\bar{b}|\sin\theta\,\hat{n}$

The vector product is also called as the cross product of two vectors.

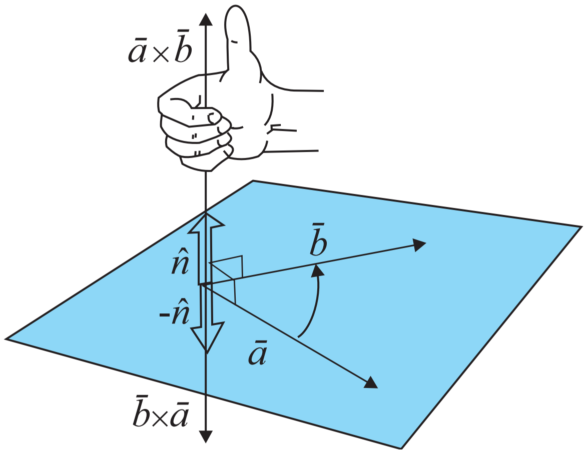


Fig.5.51

Remarks :

(i) $|\bar{a}\times\bar{b}| = |\bar{a}||\bar{b}|\sin\theta$ as $|\hat{n}| = 1$

(ii) $\bar{a}\times\bar{b}$ is perpendicular vector to the plane of $\bar{a}$ and $\bar{b}$.

(iii) The unit vector

$\hat{n}$ along $\bar{a}\times\bar{b}$ is given by $\hat{n} = \dfrac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|}$

iv) If $\bar{a}$ and $\bar{b}$ are any two coplanar (but non collinear) vectors then any vector $\bar{c}$ in the space can be given by $\bar{c} = x\bar{a} + y\bar{b} + z\left(\bar{a}\times\bar{b}\right)$ This is because $\bar{a}\times\bar{b}$ is perpendicular to both $\bar{a}$ and $\bar{b}$ and thus $\bar{a}$, $\bar{b}$ and $\bar{a}\times\bar{b}$ span the whole space.

v) If $\bar{a}, \bar{b}, \hat{n}$ form a right handed triplet, then $\bar{b}, \bar{a}, -\hat{n}$ also form a right handed triplet and

$\bar{b}\times\bar{a} = |\bar{a}||\bar{b}|\sin\theta\left(-\hat{n}\right) = -|\bar{a}||\bar{b}|\sin\theta\left(\hat{n}\right) = -\bar{a}\times\bar{b}$ .Thus vector product is anticommutative.

vi) If $\bar{a}$ and $\bar{b}$ are non zero vectors such that $\bar{a}$ is parallel to $\bar{b}$.

$\therefore \theta = 0$ i.e $\sin\theta = 0$ i.e $\bar{a}\times\bar{b} = \bar{0}$

Conversely if $\bar{a}\times\bar{b} = \bar{0}$, then either $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$ or $\sin\theta = 0$ that is $\theta = 0$.

Thus the cross product of two non zero vectors is zero only when $\bar{a}$ and $\bar{b}$ are collinear. In particular if $\bar{a} = k\bar{b}$ then $\bar{a}\times\bar{b} = k\bar{b}\times\bar{b} = k(\bar{0}) = \bar{0}$

(4) If $\bar{a},\bar{b},\bar{c}$ are any three vectors, then

(i) $\bar{a}\times(\bar{b}+\bar{c}) = \bar{a}\times\bar{b}+\bar{a}\times\bar{c}$ Left disributive law

(ii) $(\bar{b}+\bar{c})\times\bar{a} = \bar{b}\times\bar{a}+\bar{c}\times\bar{a}$ Right disributive law

viii) If $\bar{a}$ and $\bar{b}$ are any two vectors and m, n are two scalars then

(i) $m(\bar{a}\times\bar{b}) = (m\bar{a})\times\bar{b} = \bar{a}\times(m\bar{b})$

(ii) $m\bar{a}\times n\bar{b} = mn(\bar{a}\times\bar{b})$

ix) If $\hat{i},\hat{j},\hat{k}$ are the unit vectors along the co-ordinate axes then

(i) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \bar{0}$

$\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$

(ii) $\hat{j}\times\hat{i} = -\hat{k},\ \hat{k}\times\hat{j} = -\hat{i},\ \hat{i}\times\hat{k} = -\hat{j}$

(Since $\hat{i},\hat{j},\hat{k}$ from a right handed triplet)

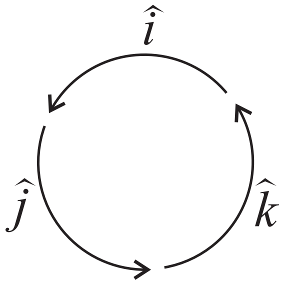


Fig.5.52

x) If $\bar{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\bar{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ are two vectors in space then

$$\bar{a}\times\bar{b} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$$

This is given using determinant by $\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Angle between two vectors: Let θ be the angle between $\bar{a}$ and $\bar{b}$ (so $0 \le \theta < \pi$),

then $|\bar{a}\times\bar{b}| = |\bar{a}|\,|\bar{b}|\sin\theta$, so $\sin\theta = \dfrac{|\bar{a}\times\bar{b}|}{|\bar{a}||\bar{b}|}$

Geometrical meaning of vector product of $\bar{a}$ and $\bar{b}$:

If $\bar{a}$ and $\bar{b}$ are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\bar{a}|$, height $|\bar{b}|\sin\theta$ and area of parallelogram

A = (Base) (Height) $= |\bar{a}|\,(|\bar{b}|\sin\theta) = |\bar{a}\times\bar{b}|$

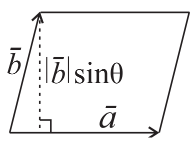


Fig.5.53

Ex. 1 Find the cross product $\bar{a}\times\bar{b}$ and verify that it is orthagonal (perpendicular) to both $\bar{a}$ and $\bar{b}$

Solved Examples

(i) $\bar{a} = \hat{i} + \hat{j} - \hat{k},\ \bar{b} = 2\hat{i} + 4\hat{j} + 6\hat{k}$ (ii) $\bar{a} = \hat{i} + 3\hat{j} - 2\hat{k}\ \bar{b} = -\hat{i} + 5\hat{k}$

Solution : (i) $\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 2 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix}\hat{i} - \begin{vmatrix} 1 & -1 \\ 2 & 6 \end{vmatrix}\hat{j} + \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix}\hat{k}$

$= \left[6-(-4)\right]\hat{i} - \left[6-(-2)\right]\hat{j} + (4-2)\hat{k} = 10\hat{i} - 8\hat{j} + 2\hat{k}$

Now $\bar{a}\times\bar{b}\cdot\bar{a} = \left(10\hat{i} - 8\hat{j} + 2\hat{k}\right)\cdot\left(\hat{i} + \hat{j} - \hat{k}\right) = 10 - 8 - 2 = 0$ and

$\bar{a}\times\bar{b}\cdot\bar{b} = \left(10\hat{i} - 8\hat{j} + 2\hat{k}\right)\cdot\left(2\hat{i} + 4\hat{j} - 6\hat{k}\right) = 20 - 32 + 12 = 0$

so $\bar{a}\times\bar{b}$ is orthogonal to both $\bar{a}$ and $\bar{b}$.

(ii) $\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 0 & 5 \end{vmatrix}\hat{i} \begin{vmatrix} 1 & -2 \\ -1 & 5 \end{vmatrix}\hat{j} + \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix}\hat{k}$

$= (15-0)\hat{i} - (5-2)\hat{j} + \left[0-(-3)\right]\hat{k} = 15\hat{i} - 3\hat{j} + 3\hat{k}$

Now $\left(\bar{a}\times\bar{b}\right)\cdot\bar{a} = \left(15\hat{i} - 3\hat{j} + 3\hat{k}\right)\cdot\left(\hat{i} + 3\hat{j} - 2\hat{k}\right) = 15 - 9 - 6 = 0,$

and $\left(\bar{a}\times\bar{b}\right)\cdot\bar{b} = \left(15\hat{i} - 3\hat{j} + 3\hat{k}\right)\cdot\left(-\hat{i} + 5\hat{k}\right) = 15 + 0 + 15 = 0,$ $\bar{a}\times\bar{b}$ is orthogonal to both $\bar{a}$ and $\bar{b}$

Ex.2: Find all vectors of magnitude $10\sqrt{3}$ that are perpendicular to the plane of $\hat{i} + 2\hat{j} + \hat{k}$ and $-\hat{i} + 3\hat{j} + 4\hat{k}$

Solution : Let $\bar{a} = \hat{i} + 2\hat{j} + \hat{k}$ and $\bar{b} = -\hat{i} + 3\hat{j} + 4\hat{k}$. Then

$$\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ -1 & 3 & 4 \end{vmatrix} = \hat{i}\,(8-3) - \hat{j}\,(4+1) + \hat{k}\,(3+2) = 5\hat{i} - 5\hat{j} + 5\hat{k} = \bar{m}\,(\text{say})$$

$$\therefore \left|\bar{a}\times\bar{b}\right| = \sqrt{(5)^2 + (-5)^2 + (5)^2} = \sqrt{3(5)^2} = 5\sqrt{3} = \left|\bar{m}\right|$$

Therefore, unit vector perpendicular to the plane of $\bar{a}$ and $\bar{b}$ is given by

$$\hat{m} = \frac{\bar{m}}{|\bar{m}|} = \frac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|} = \frac{5\hat{i} - 5\hat{j} + 5\hat{k}}{5\sqrt{3}} = \frac{\hat{i} - \hat{j} + \hat{k}}{\sqrt{3}}$$

Hence, vectors of magnitude of $10\sqrt{3}$ that are perpendicular to plane of $\bar{a}$ and $\bar{b}$ are

$$\pm 10\sqrt{3}\,\hat{m} = \pm 10\sqrt{3}\left(\frac{\hat{i} - \hat{j} + \hat{k}}{\sqrt{3}}\right), \text{ i.e. } \pm 10\left(\hat{i} - \hat{j} + \hat{k}\right).$$

Ex.3: If $\bar{u} + \bar{v} + \bar{w} = \bar{0}$, show that $\bar{u} \times \bar{v} = \bar{v} \times \bar{w} = \bar{w} \times \bar{u}.$

Solution : Suppose that $\bar{u} + \bar{v} + \bar{w} = 0.$ Then

$$\left(\bar{u} + \bar{v} + \bar{w}\right) \times \bar{v} = \bar{0} \times \bar{v}$$

$$\bar{u} \times \bar{v} + \bar{v} \times \bar{v} + \bar{w} \times \bar{v} = \bar{0}.$$

But $\bar{v} \times \bar{v} = \bar{0}$

Thus $\bar{u} \times \bar{v} + \bar{w} \times \bar{v} = \bar{0}.$

Thus $\bar{u} \times \bar{v} = -\bar{w} \times \bar{v} = \bar{v} \times \bar{w}.$

Similarly, we have $\bar{v} \times \bar{w} = \bar{w} \times \bar{u}.$

Ex 4. If $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}, \vec{b} = 2\hat{i} + \hat{j} - \hat{k}, \vec{c} = \hat{i} - 2\hat{j} + 2\hat{k}$, find $\left(\vec{a} \times \vec{b}\right) \times \vec{c}$ and $\vec{a} \times \left(\vec{b} \times \vec{c}\right)$ and hence show that $\left(\vec{a} \times \vec{b}\right) \times \vec{c} \neq \vec{a} \times \left(\vec{b} \times \vec{c}\right).$

Solution :

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = -\hat{i} + 7\hat{j} + 5\hat{k}$$

$$\left(\bar{a} \times \bar{b}\right) \times \bar{c} = \left(-\hat{i} + 7\hat{j} + 5\hat{k}\right) \times \left(\hat{i} - 2\hat{j} + 2\hat{k}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 7 & 5 \\ 1 & -2 & 2 \end{vmatrix}$$

$$= 24\hat{i} + 7\hat{j} - 5\hat{k} \qquad \text{....(1)}$$

Now, $$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & -2 & 2 \end{vmatrix} = -5\hat{j} - 5\hat{k}$$

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = (3\hat{i} - \hat{j} + 2\hat{k}) \times \left(-5\hat{j} - 5\hat{k}\right)$$

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -1 & 2\\ 0 & -5 & -5\end{vmatrix}=15(\hat{i}+\hat{j}-\hat{k}) \quad ....(2)$$

from (1) and (2), we conclude that

$$\left(\vec{a}\times\vec{b}\right)\times\vec{c}\neq\vec{a}\times\left(\vec{b}\times\vec{c}\right)$$

Ex. 5 Find the area of the triangle with vertices (1,2,0), (1,0,2), and (0,3,1).

Solution : If A = (1, 2, 0), B = (1, 0, 2) and C = (0, 3, 1), then $\overrightarrow{AB}=-2j+2k,\ \overrightarrow{AC}=-i+j+k$ and

the area of triangle ABC is $\frac{1}{2}\left|\overline{AB}\times\overline{AC}\right|$ *and* $\overline{AB}\times\overline{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & -2 & 2\\ -1 & 1 & 1\end{vmatrix}=-4\hat{i}-2\hat{j}-2\hat{k}$

$$\frac{\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|}{2}=\frac{\left|-4i-2j-2k\right|}{2}=\frac{\sqrt{16+4+4}}{2}=\frac{\sqrt{24}}{2}=\frac{2\sqrt{6}}{2}=\sqrt{6}\,sq.units$$

Ex. 6 Find the area of the parallelogram with vertices K(1, 2, 3), L(1, 3, 6), M(3, 8, 6) and N(3, 7,3)

Solution : The parallelogram is determined by the vectors $\overline{KL}=\hat{j}+3\hat{k}$ and $\overline{KN}=2\hat{i}+5\hat{j}$, so the area of parallelogram KLMN is

$$\left|\overrightarrow{KL}\times\overrightarrow{KN}\right|=\left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 1 & 3\\ 2 & 5 & 0\end{matrix}\right\|=\left|(-15)\hat{i}-(-6)\hat{j}+(-2)\hat{k}\right|=\left|-15\hat{i}+6\hat{j}-2\hat{k}\right|=\sqrt{265}\ \text{square units}$$

Ex. 7 Find $|\bar{u}\times\bar{v}|$ if

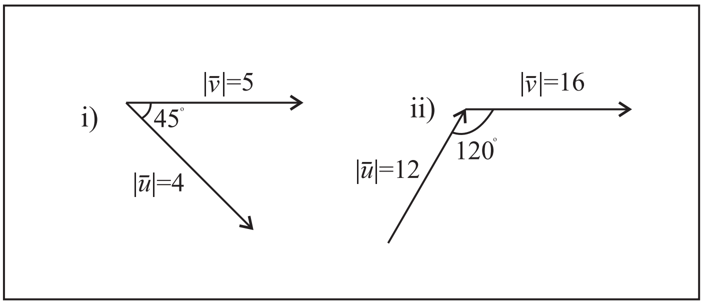


Fig 5.54

Solution : i) We have $\left|\bar{u}\times\bar{v}\right|=\left|\bar{u}\right|\left|\bar{v}\right|\sin\theta=(4)(5)\sin 45^\circ=20\frac{1}{\sqrt{2}}=10\sqrt{2}$

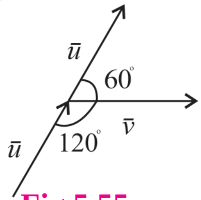


Fig 5.55

ii) If we sketch $\bar{u}$ and $\bar{v}$ starting from the same initial point, we see that the angle between them is 60°. we have $|\bar{u}\times\bar{v}| = |\bar{u}||\bar{v}|\sin\theta = (12)(16)\sin 60° = 192\,\frac{\sqrt{3}}{2} = 96\sqrt{3}$.

Ex. 8 Show that $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) = 2(\vec{a}\times\vec{b})$

Solution : Using distributive property $(a-b)\times(a+b) = a\times a + a\times b - b\times a - b\times b \quad (\because a\times b = -b\times a)$

$$= \bar{0} + \bar{a}\times\bar{b} + \bar{a}\times\bar{b} - \bar{0}\left(\because \bar{a}\times\bar{a} = \bar{0}\right)$$

$$= 2(\bar{a}\times\bar{b})$$

Ex. 9 Show that the three points with position vectors $3\hat{j}-2\hat{j}+4\hat{k},\ \hat{i}+\hat{j}+\hat{k}$ and $-\hat{i}+4\hat{j}-2\hat{k}$ respectively are collinear.

Solution : Let A, B, C be the given three points.

$\bar{a} = 3\hat{j}-2\hat{j}+4\hat{k},\ \bar{b} = \hat{i}+\hat{j}+\hat{k}$ and $\bar{c} = -\hat{i}+4\hat{j}-$ To show that points A, B, C are collinear.

Now $\overrightarrow{AB} = \bar{b}-\bar{a} = -2\hat{i}+3\hat{j}-3\hat{k},\ \ \overrightarrow{AC} = \bar{c}-\bar{a} = -4\hat{i}+6\hat{j}-6\hat{k}$.

$$\overline{AB}\times\overline{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 3 & -3 \\ -4 & 6 & -6 \end{vmatrix}$$

$$= (-18+18)\hat{i} - (12-12)\hat{j} + (-12+12)\hat{k}$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k}$$

$$= \bar{0}$$

Vectors $\overline{AB}$ and $\overline{AC}$ are collinear, but the point A is common, therefore points A, B, C are collinear.

Ex. 10 Find a unit vector perpendicular to $\overline{PQ}$ and $\overline{PR}$ where P ≡ (2,2,0), Q ≡ (0,3,5) and R ≡(5,0,3). Also find the sine of angle between $\overline{PQ}$ and $\overline{PR}$

Solution : $\overline{PQ} = \bar{q}-\bar{p} = -2\hat{i}+\hat{j}+5\hat{k}$

and $\overline{PR} = \bar{r}-\bar{p} = 3\hat{i}-2\hat{j}+3\hat{k}$

Now $|\overline{PQ}| = \sqrt{4+1+25} = \sqrt{30}$

and $|\overline{PR}| = \sqrt{9+4+9} = \sqrt{22}$

$$\therefore \overline{PQ}\times\overline{PR} = \begin{vmatrix} i & j & k \\ -2 & 1 & 5 \\ 3 & -2 & 3 \end{vmatrix}$$

$$= (3+10)\hat{i} - (-6-15)\hat{j} + (4-3)\hat{k}$$
$$= 13\hat{i} + 21\hat{j} + \hat{k}$$

$$\therefore \left|\overline{PQ} \times \overline{PR}\right| = \sqrt{169+441+1} = \sqrt{611}$$

If $\hat{n}$ is a unit vector perpendicular to $\overline{PQ}$ and $\overline{PR}$, then

$$\hat{n} = \frac{\overline{PQ} \times \overline{PR}}{\left|\overline{PQ} \times \overline{PR}\right|} = \frac{13\hat{i} + 21\hat{j} + \hat{k}}{\sqrt{611}}$$

If θ is the angle between $\overline{PQ}$ and $\overline{PR}$ then $\sin\theta = \left|\frac{\overline{PQ} \times \overline{PR}}{\left|\overline{PQ}\right|\left|\overline{PR}\right|}\right| = \frac{\sqrt{611}}{\sqrt{30}\sqrt{22}}$

Ex. 11 If $|\bar{a}| = 5$, $|\bar{b}| = 13$ and $|\bar{a} \times \bar{b}| = 25$, find $\bar{a} \cdot \bar{b}$.

Solution : Given $|\bar{a} \times \bar{b}| = 25$

$\therefore$ $|\bar{a}| \cdot |\bar{b}| \sin\theta = 25$ (θ is the angle between $\bar{a}$ and $\bar{b}$)

$5 \times 13 \sin\theta = 25$

$$\sin\theta = \frac{25}{5 \times 13} = \frac{5}{13}$$

$$\therefore \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{25}{169}} = \pm\frac{12}{13}$$

$$\therefore \bar{a} \cdot \bar{b} = |\bar{a}| \cdot |\bar{b}| \cos\theta$$

$$= 5 \times 13 \times \left(\pm\frac{12}{13}\right)$$

$$= \pm 60$$

Thus $\bar{a} \cdot \bar{b} = 60$ if $0 < \theta^{\circ} < \pi/2$ and

$\bar{a} \cdot \bar{b} = -60$ if $\theta/2 < \theta < \pi$.

Ex. 12.: Direction ratios of two lines satisfy the relation 2a-b+2c = 0 and ab+bc+ca =0. Show that the lines are perpendicular.

Solution : Given equations are 2a-b+2c = 0 i.e b = 2a+2c .... (I) and ab+bc+ca = 0....(II)

Put b = 2a+2c in equation (II), we get

a (2a+2c) + (2a+2c) c + ca = 0

$2a^2+2ac+2ac+2c^2+ac = 0$

$2a^2+5ac+2c^2 = 0$

$\therefore$ (2a+c) (a+2c) = 0

Case I : i.e. 2a+c = 0 ∴ 2a = –c...(III)

Using this equation (I) becomes b = –c + 2c = c ie. b = c..... (IV) from (III) and (IV) we get,

$\frac{a}{-\frac{1}{2}} = \frac{b}{1} = \frac{c}{1}$ Direction ratios of 1st line are i.e. $-\frac{1}{2}, 1, 1$ *i.e.* $-1, 2, 2 = \bar{p}$ (say)

Case II: i.e. a + 2c=0, ∴ a = –2c ..... (V)

Using this equation (I) becomes

b = 2(–2c) + 2c = –2c i.e. b = –2c ....(VI)

From (V) and (VI), we get

$$\frac{a}{-2} = \frac{b}{-2} = \frac{c}{1}$$

∴ Direction ratios of second line are –2, –2, 1 i.e. 2,2 –1 = $\bar{q}$ (Say)

Now $\bar{p} \cdot \bar{q} = \left(-1\hat{i} + 2\hat{j} + 2\hat{k}\right) \cdot \left(2\hat{i} + 2\hat{j} - \hat{k}\right) = -2 + 4 - 2 = 0$

∴ The lines are perpendicular.

Ex. 13 Find the direction cosines of the line which is perpendicular to the lines with direction ratios -1, 2, 2 and 0, 2, 1.

Solution : Given –1, 2,2 and 0, 2, 1 be direction ratios of lines L_1 and L_2.

Let *l*, *m*, *n* be direction cosines of line L. As line L is perpendicular to lines L_1 and L_2.

Then $-l + 2m + 2n = 0$ and $2m + n = 0$

$2m + n = 0$

∴ $2m = -n$

∴ $\frac{m}{-1} = \frac{n}{2}$

∴ ....(I) and

$-l + 2m + 2n = 0$ becomes

$-l - n + 2n = 0$

$-l + n = 0$

$l = n$

$\frac{l}{1} = \frac{n}{1}$ i.e. $\frac{1}{2} = \frac{n}{2}$ .....(II)

from (I) and (II) $\frac{l}{2} = \frac{m}{-1} = \frac{n}{2}$

The direction ratios of line L are 2, –1, 2 and the direction cosines of line L are $\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}$.

Ex. 14 If M is the foot of the perpendicular drawn from A(4, 3, 2) on the line joining the points B(2, 4, 1) and C (4, 5, 3), find the coordinates of M.

Let the point M divides BC internally in the ratio k:1

$\therefore\ M \equiv \left(\frac{4k+2}{k+1}, \frac{5k+4}{k+1}, \frac{3k+1}{k+1}\right)$ .....(I)

$\therefore$ Direction ratios of AM are

$\frac{4k+2}{k+1} - 4, \frac{5k+4}{k+1} - 3, \frac{3k+1}{k+1} - 2 = \bar{p}$ (say)

i.e. $\frac{-2}{K+1}, \frac{2k+1}{k+1}, \frac{k-1}{k+1}$ and direction ratios of BC are 4–2, 5–4, 3-1 i.e. 2, 1, 2 $= \bar{q}$ (say)

since AM is perpendicular to BC, $\bar{p} \cdot \bar{q} = 0$

i.e. $2\frac{(-2)}{k+1} + 1\frac{(2k+1)}{k+1} + 2\frac{(k-1)}{k+1} = 0$

i.e. -4+2k + 1+2k – 2 = 0

$\therefore$ 4k–5=0

$\therefore$ $k = \frac{5}{4}$

from (I) $M \equiv \left(\frac{28}{9}, \frac{41}{9}, \frac{19}{9}\right)$

Exercise 5.4

1. If $\bar{a} = 2\hat{i} + 3\hat{j} - \hat{k}, \bar{b} = \hat{i} - 4\hat{j} + 2\hat{k}$ find $(\bar{a} + \bar{b}) \times (\bar{a} - \bar{b})$

2. Find a unit vector perpendicular to the vectors $\hat{j} + 2\hat{k}$ and $\hat{i} + \hat{j}$.

3. If $\bar{a} \cdot \bar{b} = \sqrt{3}$ and $\bar{a} \times \bar{b} = 2\hat{i} + \hat{j} + 2\hat{k}$, find the angle between $\bar{a}$ and $\bar{b}$.

4. If $\bar{a} = 2\hat{i} + \hat{j} - 3\hat{k}$ *and* $\bar{b} = \hat{i} - 2\hat{j} + \hat{k}$, find a vector of magnitude 5 perpendicular to both $\bar{a}$ and $\bar{b}$.

5. Find i) $\bar{u} \cdot \bar{v}$ if $|\bar{u}| = 2, |\bar{v}| = 5, |\bar{u} \times \bar{v}| = 8$ ii) $|\bar{u} \times \bar{v}|$ if $|\bar{u}| = 10, |\bar{v}| = 2, \bar{u} \cdot \bar{v} = 12$

6. Prove that $2(\bar{a} - \bar{b}) \times 2(\bar{a} + \bar{b}) = 8(\bar{a} \times \bar{b})$

7. If $\bar{a} = \hat{i} - 2\hat{j} + 3\hat{k}, \bar{b} = 4\hat{i} - 3\hat{j} + \hat{k}$ and $\bar{c} = \hat{i} - \hat{j} + 2\hat{k}$ verify that $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$

8. Find the area of the parallelogram whose adjacent sides are the vectors $\bar{a} = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\bar{b} = \hat{i} - 3\hat{j} - 3\hat{k}$.

9. Show that vector area of a quadrilateral *ABCD* is $\frac{1}{2}\left(\overline{AC} \times \overline{BD}\right)$, where AC and BD are its diagonals.

10. Find the area of parallelogram whose diagonals are determined by the vectors
$\bar{a} = 3i - \hat{j} - 2\hat{k}$, and $\bar{b} = -\hat{i} + 3\hat{j} - 3\hat{k}$

11. If $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ are four distinct vectors such that $\bar{a} \times \bar{b} = \bar{c} \times \bar{d}$ and $\bar{a} \times \bar{c} = \bar{b} \times \bar{d}$, prove that $\bar{a} - \bar{d}$ is parallel to $\bar{b} - \bar{c}$

12. If $\bar{a} = \hat{i} + \hat{j} + \hat{k}$ and $\bar{c} = \hat{j} - \hat{k}$, find a vector $\bar{b}$ satisfying $\bar{a} \times \bar{b} = \bar{c}$ and $\bar{a} \cdot \bar{b} = 3$

13. Find $\bar{a}$, if $\bar{a} \times \hat{i} + 2\bar{a} - 5\hat{j} = 0$.

14. If $|\bar{a} \cdot \bar{b}| = |\bar{a} \times \bar{b}|$ and $\bar{a} \cdot \bar{b} < 0$, then find the angle between $\bar{a}$ and $\bar{b}$

15. Prove by vector method that $\sin(\alpha + \beta) = \sin a \cos\beta + \cos\alpha \sin\beta$.

16. Find the direction ratios of a vector perpendicular to the two lines whose direction ratios are
(i) –2,1,–1 and –3,–4,1
(ii) 1, 3, 2 and –1, 1,2

17. Prove that two vectors whose direction cosines are given by relations $al+bm+cn=0$ and $fmn+gnl+hlm=0$ are perpendicular if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$

18. If A(1, 2, 3) and B(4, 5, 6) are two points, then find the foot of the perpendicular from the point B to the line joining the origin and point A.

5.5.1 Scalar Triple Product :

We define the scalar triple product of three vectors $\bar{a}, \bar{b}, \bar{c}$ (Order is important) which is denoted by $[\bar{a}\ \bar{b}\ \bar{c}]$ and is defined as $\bar{a} \cdot (\bar{b} \times \bar{c})$

For $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\bar{b} = b_1\hat{i}, b_2\hat{j} + b_3\hat{k}$ and $\bar{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Scalar triple product is also called as box product.

Properties of scalar triple product:

Using the properties of determinant, we get following properties of scalar triple product.

(1) A cyclic change of vectors $\bar{a}, \bar{b}, \bar{c}$ in a scalar triple product does not change its value
i.e. $[\bar{a}\ \bar{b}\ \bar{c}] = [\bar{c}\ \bar{a}\ \bar{b}] = [\bar{b}\ \bar{c}\ \bar{a}]$

This follows as a cyclic change is equivalent to interchanging a pair of rows in the determinant two times.

(2) A single interchange of vectors in a scalar triple product changes the sign of its value.

i.e. $\left[\bar{a}\ \bar{b}\ \bar{c}\right] = -\left[\bar{b}\ \bar{a}\ \bar{c}\right] = -\left[\bar{c}\ \bar{b}\ \bar{a}\right] = -\left[\bar{a}\ \bar{c}\ \bar{b}\right]$

This follows as interchange of any 2 rows changes the value of determinant by sign only.

(3) If a row of determinant can be expressed as a linear combination of other rows then the determinant is zero. Using this fact we get following properties.

The scalar triple product of vectors is zero if any one of the following is true.

(i) One of the vectors is a zero vector.

(ii) Any two vectors are collinear.

(iii) The three vectors are coplanar.

(4) An interchange of 'dot' and 'cross' in a scalar triple product does not change its value

i.e. $\bar{a}\cdot\left(\bar{b}\times\bar{c}\right) = \left(\bar{a}\times\bar{b}\right)\cdot\bar{c}$

This is followed by property (1) and the commutativity of dot product.

Theorem 7 : The volume of parallelopiped with coterminus edges as $\bar{a}, \bar{b}$ and $\bar{c}$ is $\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$

Proof : Let $\overline{OA} = \bar{a}, \overline{OB} = \bar{b}$ and $\overline{OC} = \bar{c}$ be coterminus edges of parallelopiped.

Let AP be the height of the parallelopiped.

Volume of Parallelopiped = (Area of base parallelogram OBDC) (Height AP)

But AP = Scalar Projection of $\bar{a}$ on $(\bar{b} \times \bar{c}) = \dfrac{\left(\bar{b}\times\bar{c}\right)\cdot\bar{a}}{\left|\bar{b}\times\bar{c}\right|}$ $\left(\because \text{ scalar projection of } \vec{p} \text{ on } \vec{q} \text{ is } \dfrac{\bar{p}\cdot\vec{q}}{\bar{q}}\right)$

and area of parallelogram OBDC = $\left|\bar{b}\times\bar{c}\right|$

volume of parallelopiped $= \dfrac{\left(\bar{b}\times\bar{c}\right)\cdot a}{\left|\bar{b}\times\bar{c}\right|}\ \left|\bar{b}\times\bar{c}\right|$

$= \bar{a}\cdot\left(\bar{b}\times\bar{c}\right) = \left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$

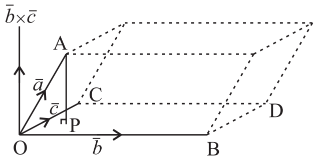


Fig . 5.56

Theorem 8. The volume of a tetrahedron with coterminus edges $\bar{a}, \bar{b}$ and $\bar{c}$ is $\dfrac{1}{6}\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$.

Proof : Let $\overline{OA} = \bar{a}, \overline{OB} = \bar{b}$ and $\overline{OC} = \bar{c}$ be coterminus edges of tetrahedron OABC.

Let AP be the height of tetrahedron

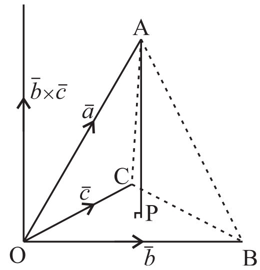


Fig. 5.57

Volume of tetrahedron $= \dfrac{1}{3}$(Area of base ΔOCB) (Height AP)

But AP = Scalar Projection of $\bar{a}$ on $(\bar{b} \times \bar{c}) = \dfrac{\left(\bar{b}\times\bar{c}\right)\cdot\bar{a}}{\left|\bar{b}\times\bar{c}\right|}$ $\left(\because \text{ scalar projection of } \vec{p} \text{ on } \vec{q} \text{ is } \dfrac{\bar{p}\cdot\vec{q}}{\bar{q}}\right)$

Area of ΔOBC = $\frac{1}{2}\,|\bar{b}\times\bar{c}|$

Volume of tetrahedron $= \frac{1}{3}\times\frac{1}{2}\,|\,\bar{b}\times\bar{c}\,|\;\frac{(\bar{b}\times\bar{c}).\bar{a}}{|\,\bar{b}\times\bar{c}\,|}$

$= \frac{1}{6}\left[(\bar{b}\times\bar{c})\cdot\bar{a}\right] = \frac{1}{6}\left[\bar{a}\;\bar{b}\;\bar{c}\right]$

5.5.2 : Vector triple product :

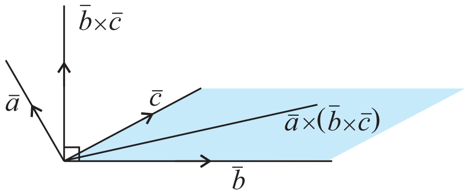


Fig. 5.58

For vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ in the space,

we define the vector triple product without proof as

$\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\,\bar{b} - (\bar{a}\cdot\bar{b})\,\bar{c}.$

Properties of vector triple product

1) $\bar{a}\times(\bar{b}\times\bar{c}) = -(\bar{b}\times\bar{c})\times\bar{a}\;(\because\;\bar{p}\times\bar{q} = -\bar{q}\times\bar{p})$

2) $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\;\vec{b} - (\vec{b}\cdot\vec{c})\;\vec{a}$

3) $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$

4) $\hat{i}\times(\hat{j}\times\hat{k}) = \vec{0}$

5) $\bar{a}\times(\bar{b}\times\bar{c})$ is linear combination of $\bar{b}$ and $\bar{c}$, hence it is coplanar with $\bar{b}$ and $\bar{c}$.

Solved Examples

Ex. 1 Find the volume of the parallelepiped determined by the vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$

(i) $\bar{a} = \hat{i}+2\hat{j}+3\hat{k},\; \bar{b} = -\hat{i}+\hat{j}+2\hat{k},\; \bar{c} = 2\hat{i}+\hat{j}+4\hat{k}$

(ii) $\bar{a} = \hat{i}+\hat{j},\; \bar{b} = \hat{j}+\hat{k},\; \bar{c} = \hat{i}+\hat{j}+\hat{k}$

Solution (i) The volume of the parallelepiped determined by $\bar{a}$, $\bar{b}$ and $\bar{c}$ is the magnitude of their scalar triple product $\bar{a}\cdot(\bar{b}\times\bar{c})$

and $\bar{a}\cdot(\bar{b}\times\bar{c}) = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix}$ = 1(4-2) -2(-4-4)+3(-1-2) = 9.

Thus the volume of the parallelepiped is 9 cubic units.

(ii) $\bar{a}\cdot(\bar{b}\times\bar{c}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$ = 0+1+0=1.

So the volume of the parallelepiped is 1 cubic unit.

Ex. 2. Find the scalar triple product $[\bar{u}\;\bar{v}\;\bar{w}]$ and verify that the vectors $\bar{u} = \hat{i}+5\hat{j}-2\hat{k}, \bar{v} = 3\hat{i}-\hat{j}$ and $\bar{w} = 5\hat{i}+9\hat{j}-4\hat{k}$ are coplanar.

Solution : $\bar{u}\cdot(\bar{v}\times\bar{w}) = \begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix} = 1\begin{vmatrix} -1 & 0 \\ 9 & -4 \end{vmatrix} - 5\begin{vmatrix} 3 & 0 \\ 5 & -4 \end{vmatrix} + (-2)\begin{vmatrix} 3 & -1 \\ 5 & 9 \end{vmatrix} = 4 + 60 - 64 = 0$

i.e. volume of the parallelepiped is 0 and thus these three vectors are coplanar.

Ex. 3. Find the vector which is orthogonal to the vector $3\hat{i}+2\hat{j}+6\hat{k}$ and is co-planar with the vectors $2\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}-\hat{j}+\hat{k}$

Solution : Let, $\bar{a}=3\hat{i}+2\hat{j}+6\hat{k}$, $\bar{b}=2\hat{i}+\hat{j}+\hat{k}$, $\bar{c}=\hat{i}-\hat{j}+\hat{k}$

Then by definition, a vector orthogonal to $\bar{a}$ and co-planar to $\bar{b}$ and $\bar{c}$ is given by

$$\bar{a}\times\left(\bar{b}\times\bar{c}\right)=\left(\bar{a}\cdot\bar{c}\right)\bar{b}-\left(\bar{a}\cdot\bar{b}\right)\bar{c}$$

$$=7\left(2\hat{i}+\hat{j}+\hat{k}\right)-14\left(\hat{i}+\hat{j}+\hat{k}\right)=21\hat{j}-7\hat{k}$$

Ex. 4. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{a}\cdot\vec{b}=1$ and $\vec{a}\times\vec{b}=\hat{j}-\hat{k}$ then prove that $\bar{b}=\hat{i}$.

Solution : As $\left(\vec{a}\times\vec{b}\right)\times\vec{a}=\left(\vec{a}\cdot\vec{a}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{a}$

$$\vec{a}\cdot\vec{a}=\left|\bar{a}\right|^2=1+1+1=3 \text{ and } \vec{a}\cdot\vec{b}=1$$

$$\therefore\left(\hat{j}-\hat{k}\right)\times\left(\hat{i}+\hat{j}+\hat{k}\right)=3\left(\vec{b}\right)-\left(\hat{i}+\hat{j}+\hat{k}\right)$$

$$i.e.\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 1 & -1\\ 1 & 1 & 1\end{vmatrix}=3\left(\vec{b}\right)-\left(\hat{i}+\hat{j}+\hat{k}\right)$$

$$\left(2\hat{i}-\hat{j}-\hat{k}\right)+\left(\hat{i}+\hat{j}+\hat{k}\right)=3\vec{b}$$

$$i.e.\ \hat{i}=\bar{b}$$

Ex. 5. Prove that : $\bar{a}\times\left(\bar{b}\times\bar{c}\right)+\bar{b}\times\left(\bar{c}\times\bar{a}\right)+\bar{c}\times\left(\bar{a}\times\bar{b}\right)=\bar{0}$

Solution : $\bar{a}\times\left(\bar{b}\times\bar{c}\right)+\bar{b}\times\left(\bar{c}\times\bar{a}\right)+\bar{c}\times\left(\bar{a}\times\bar{b}\right)$

$$\left[\left(\bar{a}\cdot\bar{c}\right)\bar{b}\left(\bar{a}\cdot\bar{b}\right)\bar{c}\right]+\left[\left(\bar{b}\cdot\bar{a}\right)\bar{c}\left(\bar{b}\cdot\bar{c}\right)\bar{a}\right]+\left[\left(\bar{c}\cdot\bar{b}\right)\bar{a}\left(\bar{c}\cdot\bar{a}\right)\bar{b}\right]$$

$$\left(\bar{a}\cdot\bar{c}\right)\bar{b}\left(\bar{a}\cdot\bar{b}\right)\bar{c}+\left(\bar{a}\cdot\bar{b}\right)\bar{c}\left(\bar{b}\cdot\bar{c}\right)\bar{a}+\left(\bar{b}\cdot\bar{c}\right)\bar{a}\left(\bar{a}\cdot\bar{c}\right)\bar{b}=\bar{0}$$

Ex. 6. Show that the points $A(2,-1,0)$ $B(-3,0,4)$, $C(-1,-1,4)$ and $D(0,-5,2)$ are non coplanar.

Solution: Let $\bar{a}=2\hat{i}-\hat{j}$, $\bar{b}=-3\hat{i}+4\hat{k}$, $\bar{c}=-\hat{i}-\hat{j}+4\hat{k}$, $\bar{c}=\hat{i}+\hat{j}+4\hat{k}$, $\bar{d}=5\hat{i}+2\hat{k}$

$$\overline{AB}=\bar{b}-\bar{c}=-5\hat{i}+\hat{i}+4\hat{k}$$

$$\overline{AC}=\bar{c}-\bar{a}=-3\hat{i}+4\hat{k}$$

$$\overline{AD}=\bar{d}-\bar{a}=-2\hat{i}-4\hat{j}+2\hat{k}$$

Consider : $\left(\overline{AB}\right)\cdot\left(\overline{AC}\right)\times\left(\overline{AD}\right)=\begin{bmatrix}-5 & 1 & 4\\ -3 & 0 & 4\\ -2 & -4 & 2\end{bmatrix}$

$= -5[0 + 16) -1\ [-6 + 8] + 4\ [12]$

$= -80 - 2 + 48$

$= 34 \neq 0$

Therefore, the points A, B, C, D are non-coplanar.

Ex. 7:. If $\overline{a}$, $\overline{b}$, $\overline{c}$ are non coplanar vectors, then show that the four points $2\overline{a}+\overline{b}, \overline{a}+2\overline{b}+\overline{c}, 4\overline{a}-2\overline{b}-\overline{c}$ and $3\overline{a}+4\overline{b}-5\overline{c}$ are coplanar.

Solution : Let 4 prints be, P ($\overline{P}$), Q ($\overline{q}$), R($\overline{r}$) and S($\overline{s}$)

$\overline{P} = 2\overline{a}+\overline{b}, q = \overline{a}+2\overline{b}+\overline{c},$

$\overline{r} = 4\overline{a}-2\overline{b}-\overline{c}, \overline{s} = 3\overline{a}+4\overline{b}-5\overline{c}$

Let us form 3 coinitial vectors

$\overline{PQ} = \overline{q}-\overline{p} = \left(\overline{a}+2\overline{b}+\overline{c}\right) - \left(2\overline{a}+\overline{b}\right) = -\overline{a}+\overline{b}+\overline{c}$

$\overline{PR} = \overline{r}-\overline{p} = \left(4\overline{a}-2\overline{b}-\overline{c}\right)-\left(2\overline{a}+\overline{b}\right) = 2\overline{a}-3\overline{b}-\overline{c}$

$\overline{PS} = \overline{s}-\overline{p} = \left(3\overline{a}+4\overline{b}-5\overline{c}\right)-\left(2\overline{a}+\overline{b}\right) = \overline{a}+3\overline{b}-5\overline{c}$

If P, Q, R, S, are coplanar then these three vectors $\overline{PQ}$, $\overline{PR}$ and $\overline{PS}$ are also coplanar this is possible only if $\overline{PQ}\cdot\left(\overline{PR}\times\overline{PS}\right) = 0$

$\overline{PQ}\cdot\left(\overline{PR}\times\overline{PS}\right) = \begin{vmatrix}-1 & 1 & 1\\ 2 & -3 & -1\\ 1 & 3 & -5\end{vmatrix}$ $= -1(15+3) -1\ (-10 + 1) +1\ (6 + 3) = -18 + 9 + 9 = 0$

Exercise 5.5

1. Find $\overline{a}\cdot\left(\overline{b}\times\overline{c}\right)$, if $\overline{a} = 3\hat{i}-\hat{j}+4\hat{k}, \overline{b} = 2\hat{i}+3\hat{j}-\hat{k}$ and $c = -5\hat{i}+2\hat{j}+3\hat{k}$

2. If the vectors $3\hat{i}+5\hat{k}, 4\hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+\hat{j}+4\hat{k}$ are to co-terminus edges of the parallelo piped, then find the volume of the parallelopiped.

3. If the vectors $-3\hat{i}+4\hat{j}-2\hat{k}, \hat{i}+2\hat{k}$ *and* $\hat{i}-p\hat{j}$ are coplanar, then find the value of p.

4. Prove that :

(i) $\left[\bar{a}\,\bar{b}+\bar{c}\;\;\bar{a}+\bar{b}+\bar{c}\right]=0$ (ii) $\left(\bar{a}+2\bar{b}-\bar{c}\right)\left[\left(\bar{a}-\bar{b}\right)\times\bar{a}-\bar{b}-\bar{c}\right]=3\left[\bar{a}-\bar{b}-\bar{c}\right]$

5. If $\bar{c}=3\bar{a}-2\bar{b}$ and $\left[\bar{a}\quad\bar{b}+\bar{c}\quad\bar{a}+\bar{b}+\bar{c}\right]=0$ then prove that $\left[\bar{a}\,\bar{b}\,\bar{c}\right]=0$

6. If $u=\hat{i}-2\hat{j}+\hat{k},\ \bar{v}=3\hat{i}+\hat{k}\ and\ \bar{w}=\hat{j}-\hat{k}$ are given vectors, then find

(i) $\left[\bar{u}+\bar{w}\right]\cdot\left[\left(\bar{w}\times\bar{r}\right)\times\left(\bar{r}\times\bar{w}\right)\right]$

7. Find the volume of a tetrahedron whose vertices are A(-1, 2, 3) B(3, -2, 1), C(2, 1, 3) and D(–1, –2, 4).

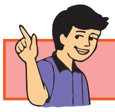
8. If $\bar{a}=\hat{i}+2\hat{j}+3,\ \bar{b}=3\hat{i}+2\hat{j}+$ and $c=2\hat{i}+\hat{j}+3$ then verify that

$\bar{a}\times\left(\bar{b}\times\bar{c}\right)=\left(\bar{a}\cdot\bar{c}\right)\bar{b}-\left(\bar{a}\cdot\bar{b}\right)\bar{c}$

9. If, $\bar{a}=\hat{i}-2\hat{j},\ \bar{b}=\hat{i}+2\hat{j}\ and\ \bar{c}=2\hat{i}+\hat{j}-2$ then find

(i) $\bar{a}\times\left(\bar{b}\times\bar{c}\right)$ (ii) $\left(\bar{a}\times\bar{b}\right)\times\bar{c}$ Are the results same? Justify.

10. Show that $\bar{a}\times\left(\bar{b}\times\bar{c}\right)+\bar{b}\times\left(\bar{c}\times\bar{a}\right)+\bar{c}\times\left(\bar{a}\times\bar{b}\right)=0$



Let's remember!

- If two vectors $\bar{a}$ and $\bar{b}$ are represented by the two adjacent sides of a parallelogram then the diagonal of the parallelogram represents $\bar{a} + \bar{b}$.
- If two vectors $\bar{a}$ and $\bar{b}$ are represented by the two adjacent sides of a triangle so that the initial point of $\bar{b}$ coincides with the terminal point of $\bar{a}$, then the vector $\bar{a} + \bar{b}$ is represented by the third side.
- Unit vector along $\bar{a}$ is denoted by $\hat{a}$ and is given by $\hat{a} = \dfrac{\bar{a}}{|\bar{a}|}$
- The vector $\overline{OP}$ where the origin O as initial point and P terminal point, is the position vector (P, V) of the point P with respect to O. $\overline{OP} = \bar{p}$.
- $\overline{AB} = \bar{b} - \bar{a}$
- Unit vector along positive X-axis, Y-axis and Z-axis denoted by $\hat{i}$, $\hat{j}$, $\hat{k}$ respectively.
- If P ≡ (x, y, z) is any point in space and O is the origin then.

$$OP = x\hat{i} + y\hat{j} + z\hat{k} \text{ and } |\overline{OP}| = \sqrt{x^2 + y^2 + z^2}$$

- Two non zero vectors $\bar{a}$ and $\bar{b}$ are said to be collinear if $\bar{a} = k\,\bar{b}$ $(k \neq 0)$
- Three non zero vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are said to be coplanar if $\bar{a} = m\bar{b} + n\bar{c}$ $(m, n \neq 0)$
- Section formula for the internal division : If R($\bar{r}$) divides the line segment joining the points A($\bar{a}$) and B($\bar{b}$) internally in the ratio m: n then $\bar{r} = \dfrac{m\bar{b} + n\bar{a}}{m+n}$
- Section formula for the external division.

  If R($\bar{r}$) is any point on the line AB such that points $A(\bar{a}), B(\bar{b})$, R$(\bar{r})$ are collinear (i.e. *A-B-R* or *R-A-B*) and $\dfrac{AR}{BR} = \dfrac{m}{n}$, and where m, n are scalars then $\bar{r} = \dfrac{m\bar{b} - n\bar{a}}{m-n}$.
- Mid Point Formula : If M($\bar{m}$) is the mid-point of the ine segment joining the points A($\bar{a}$) and B($\bar{b}$)then $\bar{m} = \dfrac{(\bar{a} + \bar{b})}{2}$.
- Centroid Formula : If G ($\bar{g}$) is the centroid of the triangle whose vertices are the point A($\bar{a}$), B($\bar{b}$),C($\bar{c}$) then $(\bar{g}) = \dfrac{\bar{a} + \bar{b} + \bar{c}}{3}$.
- If H($\bar{h}$) is incenter of Δ ABC then $\bar{h} = \dfrac{|\text{BC}|\bar{a} + |\text{AC}|\bar{b} + |\text{AB}|\bar{c}}{|\text{BC}| + |\text{AC}| + |\text{AB}|}$
- If G ($\bar{g}$) is centroiod of tetrahedron whose vertices are A($\bar{a}$), B($\bar{b}$),C($\bar{c}$) and D($\bar{d}$) then

$$\bar{g} = \frac{\bar{a}+\bar{b}+\bar{c}+\bar{d}}{4}$$

- The scalar product of two non-zero vectors and b denoted by $\bar{a}\cdot\bar{b}$, is given by $\bar{a}.\bar{b} = |\bar{a}|\ |\bar{b}|\cos\theta$

  Where θ is the angle between $\bar{a}$ and $\bar{b}$, $0 \le \theta \le \pi$
- If $\bar{a}$ is perpendicular to $\bar{b}$ then $\bar{a}\cdot\bar{b} = 0$

  $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \quad \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
- The angle θ between two non-zero vectors $\bar{a}$ and $\bar{b}$ is given by $\cos\theta = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$
- If $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\bar{a}\cdot\bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
- Scalar projection of a vector $\bar{a}$ on vector $\bar{b}$ is given $\dfrac{\bar{a}\cdot\bar{b}}{|\bar{b}|}$
- Vector projection of $\bar{a}$ on $\bar{b}$ is given by $\left(\dfrac{\bar{a}\cdot\bar{b}}{|\bar{b}|}\right)(\hat{b})$
- The vector product or cross product of two non-zero vectors $\bar{a}$ and $\bar{b}$, denoted by $\bar{a}\times\bar{b}$ is given by $\bar{a}\times\bar{b} = |\bar{a}|\cdot|\bar{b}|\ \sin\theta\ \hat{n}$

  where θ is the angle between $\bar{a}$ and $\bar{b}$, $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\bar{a}$ and $\bar{b}$.
- $\bar{a}\times\bar{b} = \bar{0}$ if and only if $\bar{a}$ and $\bar{b}$ are collinear.
- $\bar{a}\times\bar{b} = -\bar{b}\times\bar{a}$
- The angle between two vectors $\bar{a}$ and $\bar{b}$ may be given as $\dfrac{|\bar{a}\times\bar{b}|}{|\bar{a}||\bar{b}|}$
- The $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$ also

  $\hat{i}\times\hat{i} = \bar{0}$, $\hat{j}\times\hat{j} = \bar{0}$, $\hat{k}\times\hat{k} = \bar{0}$
- For a plane containing two vectors $\bar{a}$ and $\bar{b}$ the two perpendicular directions are given by $\pm(\bar{a}\times\bar{b})$.
- If $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- For $\bar{a}$ and $\bar{b}$ represent the adjacent sides of a parallelogram then its area is given by $|\bar{a}\times\bar{b}|$.

- For $\bar{a}$ and $\bar{b}$ represent the adjacent sides of a triangle then its area is given by $\frac{1}{2}\left|\bar{a}\times\bar{b}\right|$.
- If $\overline{OP}$ makes angles α, β, γ with coordinate axes, then α, β, γ are known as direction angles cos α, cos β, cos γ are known as direction cosines.

  Thus we have d.c.s of $\overline{OP}$ are $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$. i.e. l,m,n

∴ Direction cosines of $\overline{PO}$ are $-l, -m, -n$

- If l, m, n are direction cosines of a vector $\bar{r}$,

  Then (i) $l^2+m^2+n^2=1$

  (ii) $\bar{r}=\left|\bar{r}\right|\left(l\hat{i}+m\hat{j}+n\hat{k}\right)$

  (iii) $\hat{r}=l\hat{i}+m\hat{j}+n\hat{k}$

- If l, m, n are direction cosines of a vector $\bar{r}$ and a,b,c are three real numbers such that $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$ then a,b,c are called as direction ratios of vector $\bar{r}$ and

$$l=\frac{a}{\sqrt{a^2+b^2+c^2}},m=\frac{b}{\sqrt{a^2+b^2+c^2}},\text{n}=\frac{c}{\sqrt{a^2+b^2+c^2}}$$

- Scalar Triple Product (or Box Product): The dot product of $\bar{a}$ and $\bar{b}\times\bar{c}$ is called the scalar triple product of $\bar{a}$, $\bar{b}$ and $\bar{c}$. It is denoted by $\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)$ or $\left[\bar{a}\ \bar{b}\ \bar{c}\right]$.
- $\left[\bar{a}\,\bar{b}\,\bar{c}\right]=\left[\bar{b}\,\bar{c}\,\bar{a}\right]=\left[\bar{c}\,\bar{a}\,\bar{b}\right]$.
- $\left[\bar{a}\,\bar{b}\,\bar{c}\right]=-\left[\bar{b}\,\bar{a}\,\bar{c}\right]=-\left[\bar{a}\,\bar{c}\,\bar{b}\right]=-\left[\bar{c}\,\bar{b}\bar{a}\right]$
- Scalar Triple product is zero, if at least one of the vectors is a zero vector or any two vectors are collinear or all vectors are coplanar.
- Four points A($\bar{a}$), B($\bar{b}$),C($\bar{c}$)and D($\bar{d}$)are coplanar if and only if $\overline{AB}\cdot\left(\overline{AC}\times\overline{AD}\right)=0$
- The volume of the parallelopiped whose coterminus edges are represented by he vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ is given by $\left|\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)\right|$.
- The volume of the tetrahedron whose coterminus edges are given by $\bar{a}$, $\bar{b}$ and $\bar{c}$ is $\frac{1}{6}\left[\bar{a}\,\bar{b}\,\bar{c}\right]$

Miscellaneous Exercise 5

I) Select the correct option from the given alternatives :

1) If $\left|\bar{a}\right|=2$, $\left|\bar{b}\right|=3$ $\left|\bar{c}\right|=4$ then [$\bar{a}+\bar{b}$ $\bar{b}+\bar{c}$ $\bar{c}-\bar{a}$] is equal to

A) 24 B) –24 C) 0 D) 48

2) If $|\bar{a}| = 3$, $|\bar{b}| = 4$, then the value of λ for which $\bar{a} + \lambda\bar{b}$ is perpendicular to $\bar{a} - \lambda\bar{b}$,is

A) $\frac{9}{16}$ B) $\frac{3}{4}$ C) $\frac{3}{2}$ D) $\frac{4}{3}$

3) If sum of two unit vectors is itself a unit vector, then the magnitude of their difference is

A) $\sqrt{2}$ B) $\sqrt{3}$ C) 1 D) 2

4) If $|\bar{a}| = 3$, $|\bar{b}| = 5$, $|\bar{c}| = 7$ and $\bar{a} + \bar{b} + \bar{c} = 0$, then the angle between $\bar{a}$ and $\bar{b}$ is

A) $\frac{\pi}{2}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{4}$ D) $\frac{\pi}{6}$

5) The volume of tetrahedron whose vertices are (1, –6, 10), (–1, –3, 7), (5, – l, λ) and (7, –4, 7) is 11 cu. units then the value of λ is

A) 7 B) $\frac{\pi}{3}$ C) 1 D) 5

6) If α, β, γ are direction angles of a line and $\alpha = 60^\circ$, $\beta = 45^\circ$, the γ =

A) 30° or 90° B) 45° or 60° C) 90° or 30° D) 60° or 120°

7) The distance of the point (3, 4, 5) from Y- axis is

A) 3 B) 5 C) $\sqrt{34}$ D) $\sqrt{41}$

8) The line joining the points (–2, 1, –8) and (a, b, c) is parallel to the line whose direction ratios are 6, 2, 3. The value of a, b, c are

A) 4, 3, –5 B) $1, 2, \frac{-13}{2}$ C)10, 5, –2 D) 3, 5, 11

9) If cos α, cos β, cos γ are the direction cosines of a line then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is

A) 1 B) 2 C) 3 D) 4

10) If *l*, *m*, *n* are direction cosines of a line then $l\hat{i} + m\hat{j} + n\hat{k}$ is

A) null vector B) the unit vector along the line

C) any vector along the line D) a vector perpendicular to the line

11) If $|\bar{a}| = 3$ and $-1 \leq k \leq 2$, then $|k\vec{a}|$ lies in the interval

A) [0, 6] B) [–3, 6] C) [3, 6] D) [1,2]

12) Let α, β, γ be distinct real numbers. The points with position vectors $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$, $\beta\hat{i} + \gamma\hat{j} + \alpha\hat{k}$, $\gamma\hat{i} + \alpha\hat{j} + \beta\hat{k}$

A) are collinear B) form an equilateral triangle

C) form a scalene triangle D) form a right angled triangle

13) Let $\overline{p}$ and $\overline{q}$ be the position vectors of P and Q respectively, with respect to O and $|\overline{p}|$ = p, $|\overline{q}|$ = q. The points R and S divide PQ internally and externally in the ratio 2 : 3 respectively. If OR and OS are perpendicular then.

A) $9p^2 = 4q^2$ B) $4p^2 = 9q^2$ C) 9p = 4q D) 4p = 9q

14) The 2 vectors $\hat{j} + \hat{k}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represents the two sides AB and AC, respectively of a ΔABC. The length of the median through A is

A) $\frac{\sqrt{34}}{2}$ B) $\frac{\sqrt{48}}{2}$ C) $\sqrt{18}$ D) None of these

15) If $\vec{a}$ and $\vec{b}$ are unit vectors, then what is the angle between $\vec{a}$ and $\vec{b}$ for $\sqrt{3}\overline{a} - \overline{b}$ to be a unit vector ?

A) 30º B) 45º C) 60º D) 90º

16) If θ be the angle between any two vectors $\overline{a}$ and $\overline{b}$, then $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$, when θ is equal to

A) 0 B) $\frac{\pi}{4}$ C) $\frac{\pi}{2}$ D) π

17) The value of $\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j})$

A) 0 B) –1 C) 1 D) 3

18) Let a, b, c be distinct non-negative numbers. If the vectors $a\hat{i} + a\hat{j} + c\hat{k}$, $\hat{i} + \hat{k}$ and $c\hat{i} + c\hat{j} + b\hat{k}$ lie in a plane, then c is

A) The arithmetic mean of a and b B) The geometric mean of a and b

C) The harmonic man of a and b D) 0

19) Let $\overline{a} = \hat{i} \quad \hat{j}$, $\overline{b} = \hat{j} \quad \hat{k}$, $\overline{c} = \hat{k} \quad \hat{i}$. If $\overline{d}$ is a unit vector such that $\overline{a} \cdot \overline{d} = 0 = [\overline{b} \quad \overline{c} \quad \overline{d}]$, then $\overline{d}$ equals.

A) $\pm\frac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}}$ B) $\pm\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ C) $\pm\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ D) $\pm\hat{k}$

20) If $\vec{a} \quad \vec{b} \quad \vec{c}$ are non coplanar unit vectors such that $\overline{a} \times (\overline{b} \times \overline{c}) = \frac{(\overline{b} + \overline{c})}{\sqrt{2}}$ then the angle between $\vec{a}$ and $\vec{b}$ is

A) $\frac{3\pi}{4}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{2}$ D) π

II Answer the following :

1) ABCD is a trapezium with AB parallel to DC and DC = 3AB. M is the mid-point of DC, $\overrightarrow{AB} = \bar{p}$ and $\overrightarrow{BC} = \bar{q}$. Find in terms of $\bar{p}$ and $\bar{q}$.

i) $\overrightarrow{AM}$ ii) $\overrightarrow{BD}$ iii) $\overrightarrow{MB}$ iv) $\overrightarrow{DA}$

2) The points A, B and C have position vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ respectively. The point P is midpoint of AB. Find in terms of $\bar{a}$, $\bar{b}$ and $\bar{c}$ the vector $\overrightarrow{PC}$

3) In a pentagon ABCDE

Show that $\overline{AB} + \overline{AE} + \overline{BC} + \overline{DC} + \overline{ED} = 2\overline{AC}$

4) If in parallelogram ABCD, diagonal vectors are $\overline{AC} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\overrightarrow{BD} = -6\hat{i} + 7\hat{j} - 2\hat{k}$, then find the adjacent side vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$

5) If two sides of a triangle are $\hat{i} + 2\hat{j}$ and $\hat{i} + \hat{k}$, then find the length of the third side.

6) If $|\bar{a}| = |\bar{b}| = 1$ $\bar{a} \cdot \bar{b} = 0$ and $\bar{a} + \bar{b} + \bar{c} = 0$ then find $|\bar{c}|$

7) Find the lengths of the sides of the triangle and also determine the type of a triangle.

i) A(2, –1, 0), B(4, 1, 1,), C(4, –5, 4) ii) L(3, –2, –3), M(7, 0,1), N (1, 2, 1)

8) Find the component form of if $\bar{a}$ if

i) It lies in YZ plane and makes 60º with positive Y-axis and $|\bar{a}| = 4$

ii) It lies in XZ plane and makes 45º with positive Z-axis and $|\bar{a}| = 10$

9) Two sides of a parallelogram are $3\hat{i} + 4\hat{j} - 5\hat{k}$ and $-2\hat{j} + 7\hat{k}$. Find the unit vectors parallel to the diagonals.

10) If *D*, *E*, *F* are the mid-points of the sides *BC*, *CA*, *AB* of a triangle *ABC*, prove that

$\overline{AD} + \overline{BE} + \overline{CF} = \bar{0}$

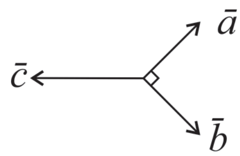


Fig.5.59

11) Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point (2, 4)

12) Express the vector $\hat{i} + 4\hat{j} - 4\hat{k}$ as a linear combination of the vectors $2\hat{i} - \hat{j} + 3\hat{k}$, $\hat{i} - 2\hat{j} + 4\hat{k}$ and $-\hat{i} + 3\hat{j} - 5\hat{k}$

13) If $\overline{OA} = \bar{a}$ and $\overline{OB} = \bar{b}$ then show that the vector along the angle bisector of angle AOB is given by $\bar{d} = \lambda\left(\frac{\bar{a}}{|\bar{b}|} + \frac{\bar{b}}{|\bar{b}|}\right)$

14) The position vectors f three consecutive vertices of a parallelogram are $\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + 3\hat{j} + 5\hat{k}$ and $7\hat{i} + 9\hat{j} + 11\hat{k}$ Find the position vector of the fourth vertex.

15) A point P with $p.v. \dfrac{-14\hat{i}+39\hat{j}+28\hat{k}}{5}$ divides the line joining A(–1, 6, 5) and B in the ratio 3:2 then find the point B.

16) Prove that the sum of the three vectors determined by the medians of a triangle directed from the vertices is zero.

17) ABCD is a parallelogram E, F are the mid points of BC and CD respectively. AE, AF meet the diagonal BD at Q and P respectively. Show that P and Q trisect DB.

18) If aBC is a triangle whose orthocenter is P and the circumcenter is Q, then prove that

$\overline{PA} + \overline{PC} + \overline{PB} = 2\,\overline{PQ}$

19) If P is orthocenter, Q is circumcenter and G is centroid of a triangle ABC, then prove that

$\overline{QP} = 3\,\overline{QG}$

20) In a triangle OAB, E is the midpoint of BO and D is a point on AB such that AD: DB = 2:1. If OD and AE intersect at P, determine the ratio OP:PD using vector methods.

21) Dot-product of a vector with vectors $3\hat{i} - 5\hat{k}$, $2\hat{i} + 7\hat{j}$ and $\hat{i} + \hat{j} + \hat{k}$ are respectively –1, 6 and 5. Find the vector.

22) If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$, then find the value of $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$

23) If a parallelogram is constructed on the vectors $\vec{a} = 3\bar{p} - \bar{q}$, $\bar{b} = \bar{p} + 3\bar{q}$ and $|\bar{p}| = |\bar{q}| = 2$ and angle between $\bar{p}$ and $\bar{q}$ is $\pi/3$ show that the ratio of the lengths of the sides is $\sqrt{7} : \sqrt{13}$

24) Express the vector $\vec{a} = 5\hat{i} - 2\hat{j} + 5\hat{k}$ as a sum of two vectors such that one is parallel to the vector $\bar{b} = 3\hat{i} + \hat{k}$ and other is perpendicular to $\bar{b}$.

25) Find two unit vectors each of which makes equal angles with $\bar{u}$, $\bar{v}$ and $\bar{w}$.

$\bar{u} = 2\hat{i} + \hat{j} - 2\hat{k}$, $\bar{v} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\bar{w} = 2\hat{i} - 2\hat{j} + \hat{k}$

26) Find the acute angles between the curves at their points of intersection. $y = x^2$, $y = x^3$

27) Find the direction cosines and direction angles of the vector.

i) $2\hat{i} + \hat{j} + 2\hat{k}$ (ii) $(1/2)\,\hat{i} + \hat{j} + \hat{k}$

28) Let b = $4\hat{i} + 3\hat{j}$ and $\vec{c}$ be two vectors perpendicular to each other in the XY-plane. Find vectors in the same plane having projection 1 and 2 along $\bar{b}$ and $\bar{c}$, respectively, are given y.

29) Show that no line in space can make angle π/6 and π /4 with X- axis and Y-axis.

30) Find the angle between the lines whose direction cosines are given by the equation 6mn–2nl+5lm=0, 3l+m+5n=0

31) If Q is the foot of the perpendicular from P(2,4,3) on the line joining the points A(1,2,4) and B(3,4,5), find coordinates of Q.

32) Show that the area of a triangle ABC, the position vectors of whose vertices are a, b and c is $\frac{1}{2}\left[\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right]$

33) Find a unit vector perpendicular to the plane containing the point (a, 0, 0), (0, b, 0), and (0, 0, c). What is the area of the triangle with these vertices?

34) State whether each expression is meaningful. If not, explain why ? If so, state whether it is a vector or a scalar.

(a) $\bar{a}\cdot(\bar{b}\times\bar{c})$ (b) $\bar{a}\times(\bar{b}\cdot\bar{c})$

(c) $\bar{a}\times(\bar{b}\times\bar{c})$ (d) $\bar{a}\cdot(\bar{b}\cdot\bar{c})$

(e) $(\bar{a}\cdot\bar{b})\times(\bar{c}\cdot\bar{d})$ (f) $(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d})$

(g) $(\bar{a}\cdot\bar{b})\cdot\bar{c}$ (h) $(\bar{a}\cdot\bar{b})\bar{c}$

(i) $(|\bar{a}|)(\bar{b}\cdot\bar{c})$ (j) $\bar{a}\cdot(\bar{b}+\bar{c})$

(k) $\bar{a}\cdot\bar{b}+\bar{c}$ (l) $|\bar{a}|\cdot(\bar{b}+\bar{c})$

35. Show that, for any vectors $\bar{a},\bar{b},\bar{c}$

$$(\bar{a}+\bar{b}+\bar{c})\times\bar{c}+(\bar{a}+\bar{b}+\bar{c})\times\bar{b}+(\bar{b}+\bar{c})\times\bar{a}=2\bar{a}\times\bar{c}$$

36. Suppose that $\bar{a}=0$.

(a) If $\bar{a}\cdot\bar{b}=\bar{a}\cdot\bar{c}$ then is $\bar{b}=\bar{c}$?

(b) If $\bar{a}\times\bar{b}=\bar{a}\times\bar{c}$ then is $\bar{b}=\bar{c}$?

(c) If $\bar{a}\cdot\bar{b}=\bar{a}\cdot\bar{c}$ and $\bar{a}\times\bar{b}=\bar{a}\times\bar{c}$ then is $\bar{b}=\bar{c}$?

37. If A(3, 2, -1), B(-2, 2, -3), C(3, 5, -2), D(-2, 5, -4) then (i) verify that the points are the vertices of a parallelogram and (ii) find its area.

38. Let A, B, C, D be any four points in space. Prove that

$$\left|\overline{AB}\times\overline{CD}+\overline{BC}\times\overline{AD}+\overline{CA}\times\overline{BD}\right| = 4 \text{ (area of } \Delta ABC)$$

39. Let $\hat{a},\hat{b},\hat{c}$ be unit vectors such that $\hat{a}\cdot\hat{b}=\hat{a}\cdot\hat{c}=0$ and the angle between $\hat{b}$ and $\hat{c}$ be $\pi/6$.

Prove that $\hat{a}=\pm 2\left(\hat{b}\times\hat{c}\right)$

40. Find the value of 'a' so that the volume of parallelopiped a formed by $\hat{i}+\hat{j}+\hat{k}+a\hat{k}$ and $a\hat{j}+\hat{k}$ becomes minimum.

41. Find the volume of the parallelepiped spanned by the diagonals of the three faces of a cube of side a that meet at one vertex of the cube.

42. If $\bar{a},\bar{b},\bar{c}$ are three non-coplanar vectors, then show that

$$\frac{\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)}{\left(\bar{c}\times\bar{a}\right)\cdot\bar{b}}+\frac{\bar{b}\cdot\left(\bar{a}\times\bar{c}\right)}{\left(\bar{c}\times\bar{a}\right)\cdot\bar{b}}=0$$

43. Prove that $\left(\bar{a}\times\bar{b}\right)\cdot\left(\bar{c}\times\bar{d}\right)\begin{vmatrix}\bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c}\\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d}\end{vmatrix}$

44. Find the volume of a parallelopiped whose coterminus edges are represented by the vector $\hat{j}+\hat{k}$. $\hat{i}+\hat{k}$ and $\hat{i}+\hat{j}$. Also find volume of tetrahedron having these coterminous edges.

45. Using properties of scalar triple product, prove that $[\bar{a}+\bar{b} \quad \bar{b}+\bar{c} \quad \bar{c}+\bar{a}] = 2[\bar{a} \quad \bar{b} \quad \bar{c}]$.

46) If four points A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) and D($\bar{d}$) are coplanar then show that $[\bar{a} \; \bar{b} \; \bar{d}]+[\bar{b} \; \bar{c} \; \bar{d}]+[\bar{c} \; \bar{a} \; \bar{d}]=[\bar{a} \; \bar{b} \; \bar{c}]$

47) If $\bar{a}$ $\bar{b}$ and $\bar{c}$ are three non coplanar vectors, then $\left(\bar{a}+\bar{b}+\bar{c}\right)\cdot\left[\left(\bar{a}+\bar{b}\right)\times\left(\bar{a}+\bar{c}\right)\right]=-\left[\bar{a} \quad \bar{b} \quad \bar{c}\right]$

48) If in a tetrahedron, edges in each of the two pairs of opposite edges are perpendicular, then show that the edges in the third pair are also perpendicular.

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